Lattices

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Mathematical Foundations of Cryptography – WT 2020/21

Outline

Literature

The slides are based on the following sources

- **An Introduction to Mathematical Cryptography**, Hoffstein, Jeffrey, Pipher, Jill, Silverman, J.H.
- **A Decade of Lattice Cryptography**, Chris Peikert
- **The LLL Algorithm**, Phong Q. Nguyen, Brigitte Vallée (Eds.)

Many graphics are based on graphics from Maria Eichlseder.

Lattice-Based Cryptography

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- Algorithmic simplicity, efficiency, and parallelism.
- Strong security guarantees from worst-case hardness.
- Construction of versatile and powerful cryptographic objects
  - Fully Homomorphic Encryption
  - Attribute-Based Encryption

Vector Spaces

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- A linear combination of the vectors $v_1, \dots, v_k$ is any vector of the form

$$w = \alpha_1 v_1 + \cdots + \alpha_k v_k, \text{ with } \alpha_1, \dots, \alpha_k \in \mathbb{R}.$$

  The collection of all such linear combinations is called the span of $\{v_1, \dots, v_k\}$.
  e.g.

$$w = 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Vector Spaces

- A set of vectors $v_1, \ldots, v_k \in V$ is linearly independent if

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \cdots = \alpha_k = 0.$$

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- There always exists a basis for V.
- Any two bases for V have the same number elements.

Length and Angle 1

- The dot product of $v = (x_1, \ldots, x_m), w = (y_1, \ldots, y_m) \in V$ is the quantity

$$v \cdot w = x_1 y_1 + \cdots + x_m y_m.$$

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- The length, or Euclidean norm, of v is the quantity

$$\|v\| = \sqrt{x_1^2 + \cdots + x_m^2}.$$

$$v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; \|v\| = ?$$

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- Let α be the angle between v and w, then

$$v \cdot w = \|v\| \|w\| \cos(\alpha).$$

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$\mathrm{vol}(v_1, v_2) = \sqrt{\det G} = \sqrt{25} = 5$

Gram-Schmidt Algorithm

Theorem (Gram-Schmidt Algorithm)

Let $v_1, \ldots, v_n$ be a basis for a vector space $V \subset \mathbb{R}^m$. The following algorithm creates an orthogonal basis $v_1^*, \ldots, v_n^*$ for V:

$v_1^* \leftarrow v_1$

for $i = 2..n$ **do**

  for $j = 1..i-1$

$$\mu_{i,j} \leftarrow \frac{v_i \cdot v_j^*}{\|v_j^*\|^2}$$

$$v_i^* = v_i - \sum_{j=1}^{i-1} \mu_{i,j} v_j^*$$

Definition and Properties

Lattices

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- an additive subgroup
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In other words, let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a set of linearly independent vectors. The lattice generated by $v_1, \ldots, v_n$ is the set of linear combinations of $v_1, \ldots, v_n$ with coefficients in $\mathbb{Z}$,

$$L = \{a_1 v_1 + \cdots + a_n v_n \; : \; a_1, \ldots, a_n \in \mathbb{Z}\}.$$

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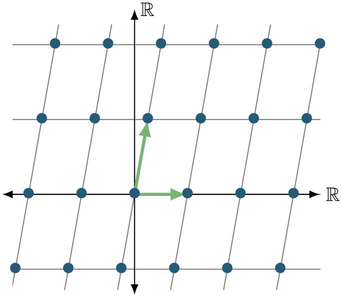
Example:

$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1/4 \\ \sqrt{2} \end{pmatrix}$



Fundamental Domains

Definition (Fundamental Domain)

Let L be a lattice of dimension n and let $v_1, \ldots, v_n$ be a basis for L. The fundamental domain is the set

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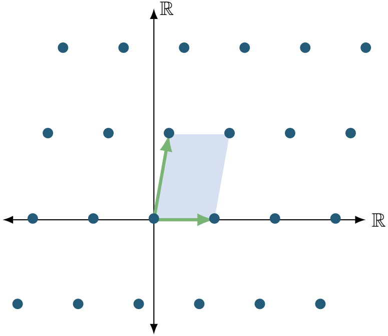

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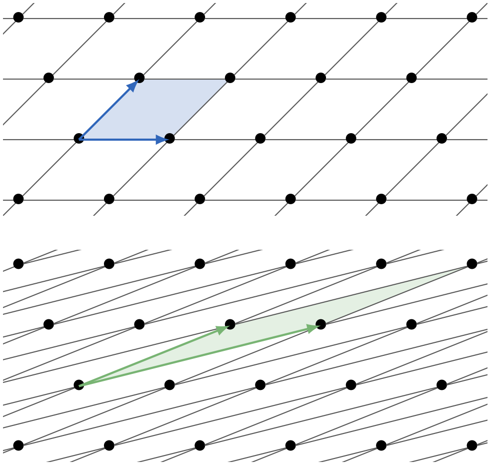
Therefore,

$$\mathrm{vol}(L) = \sqrt{\det G} = \sqrt{2}$$

Same Lattice?

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\boldsymbol{v}'_1 = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \boldsymbol{v}'_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$



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$$G = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 6 & 8 \end{pmatrix}.$$

$$G' = \begin{pmatrix} 8 & 2 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 68 & 44 \\ 44 & 29 \end{pmatrix}.$$

Therefore $V = \sqrt{G} = \sqrt{36} = 6 = \sqrt{36} = \sqrt{G'} = V'$.

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Proposition

Every fundamental domain for a given lattice L has the same volume.

Short Vectors in Lattices

Computational Problems

$\lambda_1(L)$... length of shortest nonzero vector in L.

- **Shortest Vector Problem (SVP):** Find a shortest nonzero vector v in L, i.e. $\|v\| = \lambda_1(L)$.
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and given the vector $w = (2, 3)^T$. What is a shortest nonzero vector of L? Which vector is closest to w?

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$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

How long is the shortest vector?

Theorem (Minkowski's Theorem)

Let $L \subset \mathbb{R}^n$ be a lattice of dimension n. Let $S \subset \mathbb{R}^n$ be convex, closed and symmetric. Suppose that $\mathrm{vol}(S) \geq 2^n \,\mathrm{vol}(L)$, then

$$S \cap L \supsetneq \{0\}.$$

S... hypercube in $\mathbb{R}^n$ centered at 0 with length $2\,\mathrm{vol}(L)^{1/n}$, then $\mathrm{vol}(S) = 2^n \,\mathrm{vol}(L)$.
Applying Minkowski's theorem leads to:

Corollary (Hermite's Theorem)

Every lattice L of dimension n contains a nonzero $v \in L$ satisfying

$$\|v\| \leq \sqrt{n}\,\mathrm{vol}(L)^{\frac{1}{n}}.$$

Lattice Reduction Algorithms

Babai's Closest Vertex Algorithm

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3. Return $v = a_1 v_1 + \cdots + a_n v_n$.

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Orthogonality Defects

Definition (Hadamard Ratio)

We define the Hadamard ratio of the basis $B = \{v_1, \ldots, v_n\}$ to be the quantity

$$H(B) = \left(\frac{\text{vol}(L)}{\|v_1\| \cdots \|v_n\|} \right)^{\frac{1}{n}} \in (0, 1].$$

(the closer to 1, the more orthogonal)

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$$H(B) = \left(\frac{6}{\sqrt{9}\sqrt{8}} \right)^{\frac{1}{2}} \approx 0.84.$$

Recap

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SVP: Minkowski's and Hermite's Theorem

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Lattice:	Basis, Fundamental Domain, Volume
SVP:	Minkowski's and Hermite's Theorem
Reduction:	Babai's Algorithm

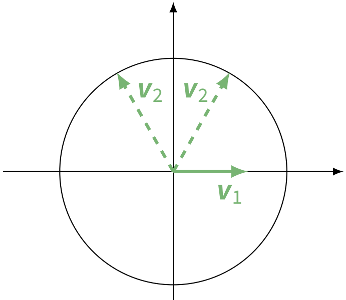
Lagrange-Reduced

Definition (Lagrange-reduced)

Let L be a two-dimensional lattice. A basis (v_1, v_2) of L is said to be Lagrange-reduced if and only if

$$\|v_1\| \leq \|v_2\| \quad \text{and} \quad |v_1 \cdot v_2| \leq \frac{\|v_1\|^2}{2}.$$

Optimal: $\lambda_1(L) = \|v_1\|$



Lagrange's Reduction Algorithm

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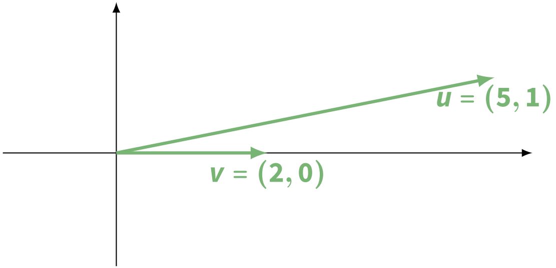
```
if ‖u‖ > ‖v‖ then
    sawp u and v
while ‖v‖ > ‖u‖ do
    r ← u − qv  where q = ⌊ (u · v) / ‖v‖² ⌉
    u ← v
    v ← r
return (u, v)
```

Lagrange Reduction: Example

Input: $v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, u = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$



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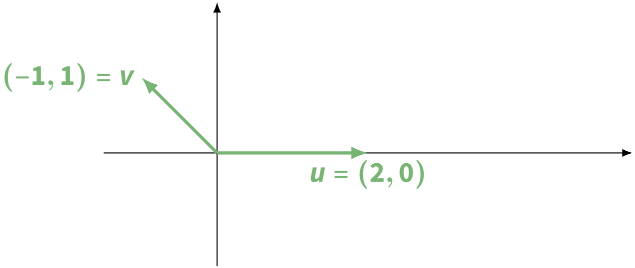

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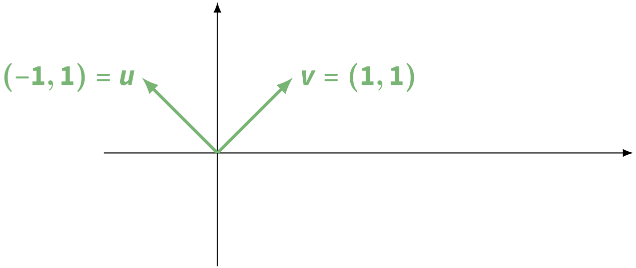

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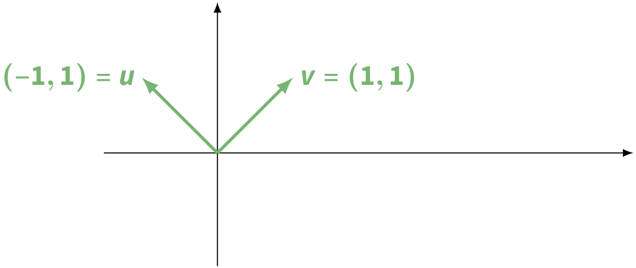

Task: Solve SVP for the lattice generated by

$$v_1 = (66586820, 65354729)^T, v_2 = (6513996, 6393464)^T.$$