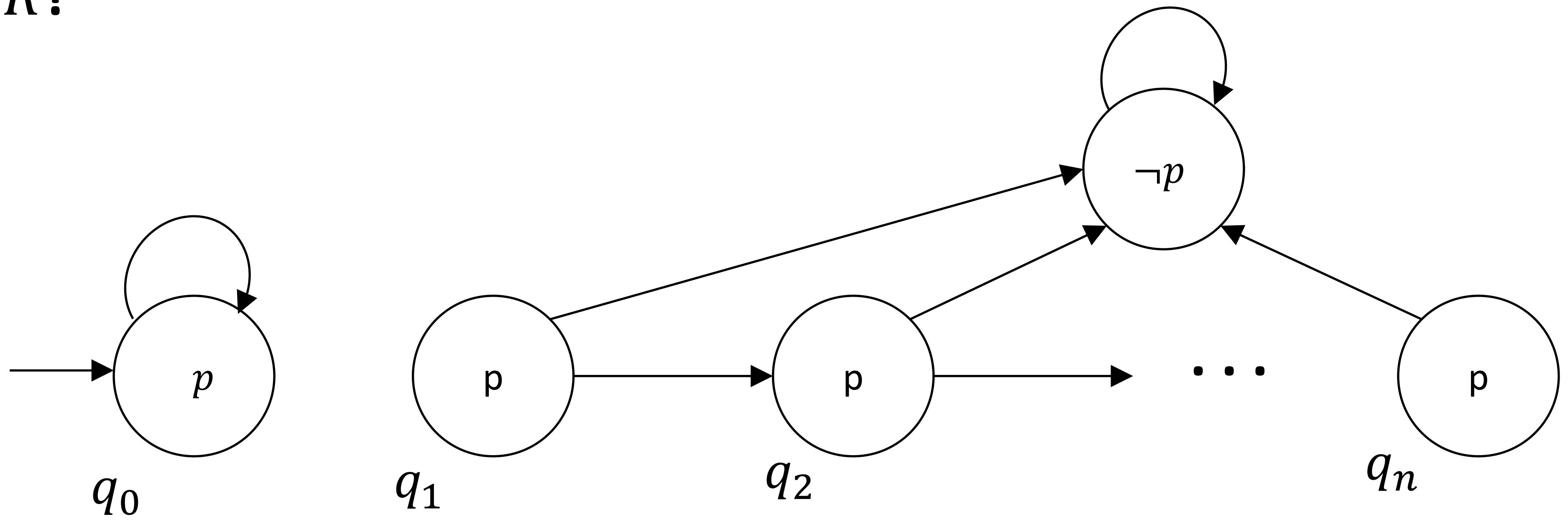
Model Checking with Inductive Invariants

Problems with *k*-induction

Problem: Sometimes *k* is very large

In the following machine, you need $k = n + 1$ to prove $\mathbf{AG}\, p$.

Idea: Automatically find better inductive invariants. Avoid many copies of R**!**

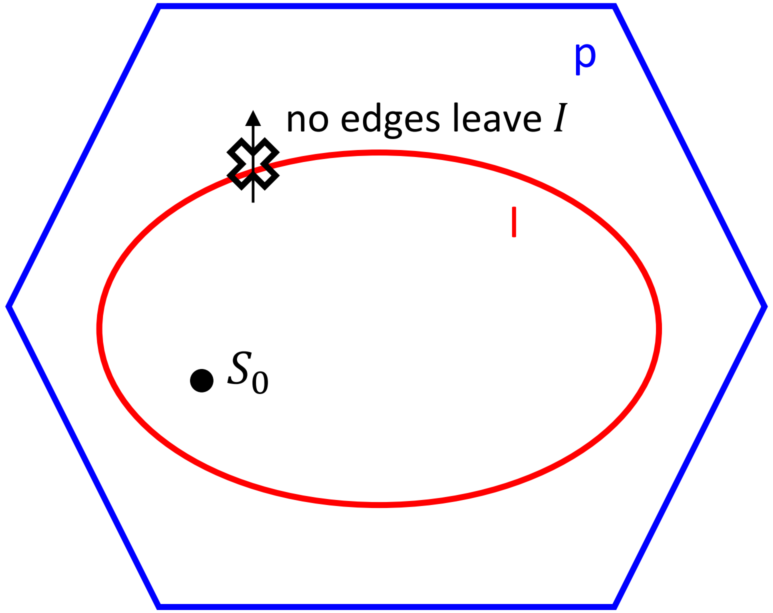


Inductive Invariant

Remember $postimage(Q) = \{ s' \mid \exists s. R(s,s')\}$ (see Chapter 5).

Definition. $I \subseteq S$ is an inductive invariant for $\mathrm{AG}\, p$ if

1. $S_0 \subseteq I$
2. $postimage(I) \subseteq I$
3. $\forall s \in I. s \vDash p$



If there is an inductive invariant for $\mathrm{AG}\, p$, then $\mathrm{AG}\, p$ holds.

In formulas:

1. $S_0 \rightarrow I$
2. $I \wedge R \rightarrow I'$
3. $\mathrm{I} \rightarrow p$

Inductive Invariant

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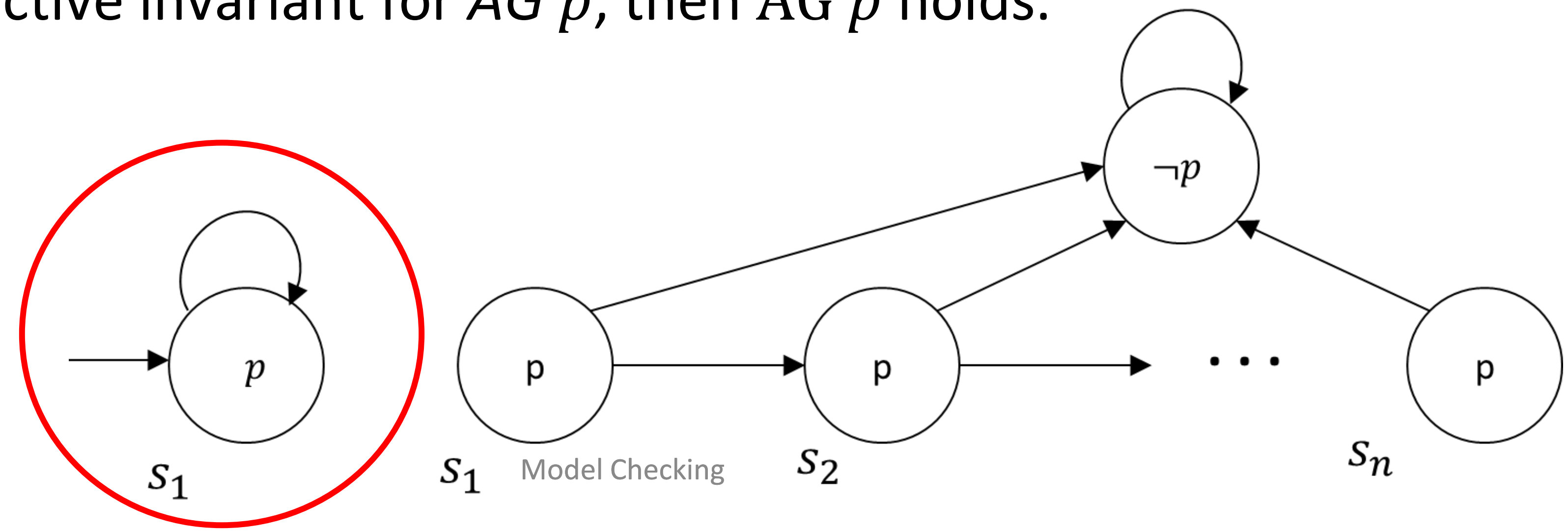
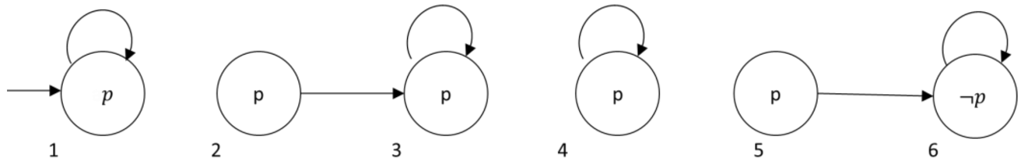
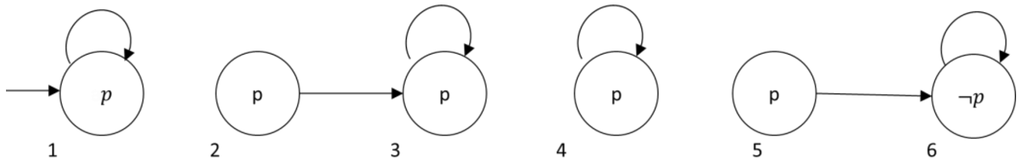
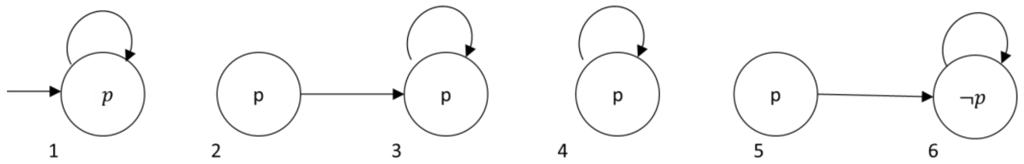
If there is an inductive invariant for $AG\ p$, then $\text{AG}\ p$ holds.

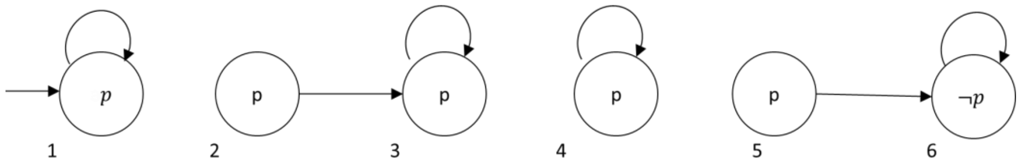


Multiple Invariants

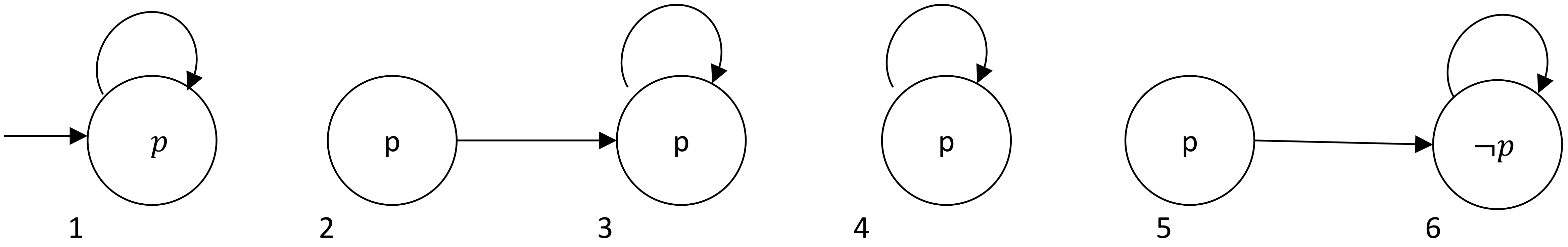








Strongest & Weakest Invariant



1. Smallest (strongest) invariant consists of reachable states
2. Largest (weakest) invariant consists of states that cannot reach $\neg p$

Cf. two hypothetical model checking algorithms:

1. Start with initial states, keep adding successors until nothing changes or you reach $\neg p$
2. Start with $\neg p$, keep adding predecessors until nothing changes or you reach an initial state

Model Checking with Craig Interpolants

Ken McMillan, 2003

2010 CAV Award: “has significantly influenced both academic research and industrial practice, and has dramatically changed the scale of systems that can be analyzed by model checking.”

Kenneth McMillan

Interpolants as Inductive Invariants

- BMC finds bugs (and absence of bugs up to k steps)
- How to Show Correctness?
  - k-induction
  - Interpolants

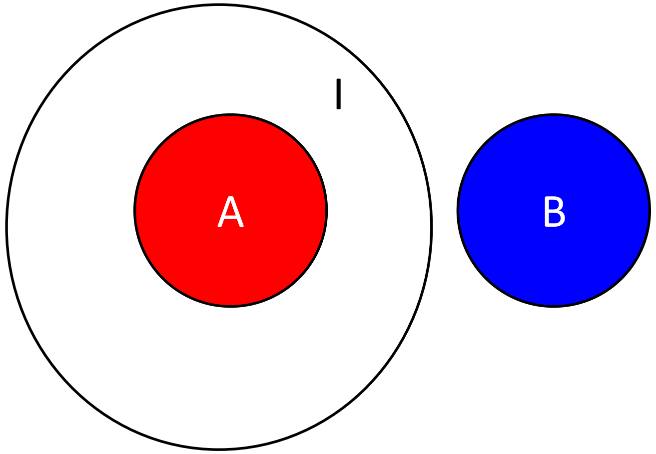
- Find **Interpolants** I such that
  - States reachable in k steps are in I
  - no bad states are in I
- Interpolants are (good) overapproximation of post-image computation

Interpolant

William Craig, 1957

Definition. Given formulas A, B such that $A \wedge B = \bot$, an interpolant is a formula I such that

1. $A \rightarrow I$
2. $I \wedge B \equiv \bot$
3. I only uses symbols that occur both in A and in B



Example. Let

$A = (a_1 \vee \neg a_2) \wedge (\neg a_1 \vee \neg a_3) \wedge a_2$,

$B = (\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4$.

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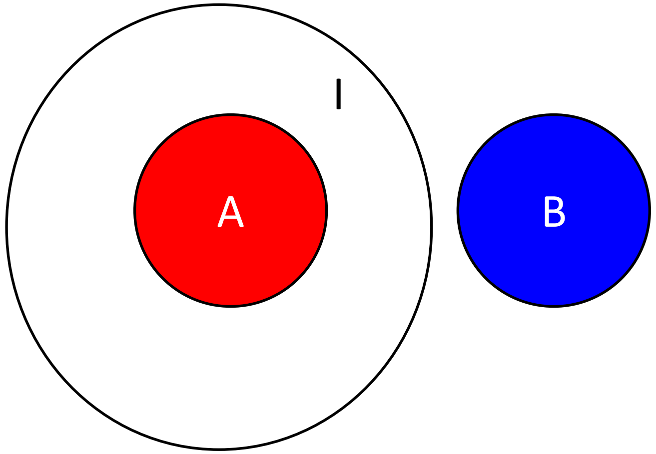

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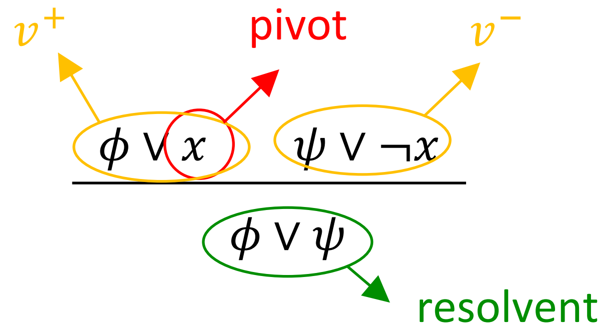
$B = (\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4$.

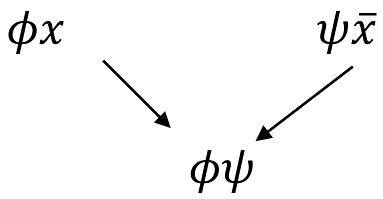
$A \wedge B$ is not satisfiable.

$\neg a_3 \wedge a_2$ is an interpolant:

1. $\big((a_1 \vee \neg a_2) \wedge (\neg a_1 \vee \neg a_3) \wedge a_2\big) \rightarrow (\neg a_3 \wedge a_2)$
2. $(\neg a_3 \wedge a_2) \wedge \big((\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4\big) \equiv \bot$
3. a_2 and a_3 occur in A and in B

Resolution (Chap 9)





Interpolants from Resolution Proofs

For clause C, $C|B$ is obtained by removing literals not in B

Algorithm. Go through resolution proof top-down.

1. If leaf v is labeled $C \in A$, then $Itp(v) = C|B$
2. If leaf v is labeled $C \in B$, then $Itp(v) = \top$
3. If node v has pivot variable $x \in B$ then $Itp(v) = Itp(v^{+}) \wedge Itp(v^{-})$
4. If node v has pivot variable $x \notin B$ then $Itp(v) = Itp(v^{+}) \vee Itp(v^{-})$

Interpolation Example

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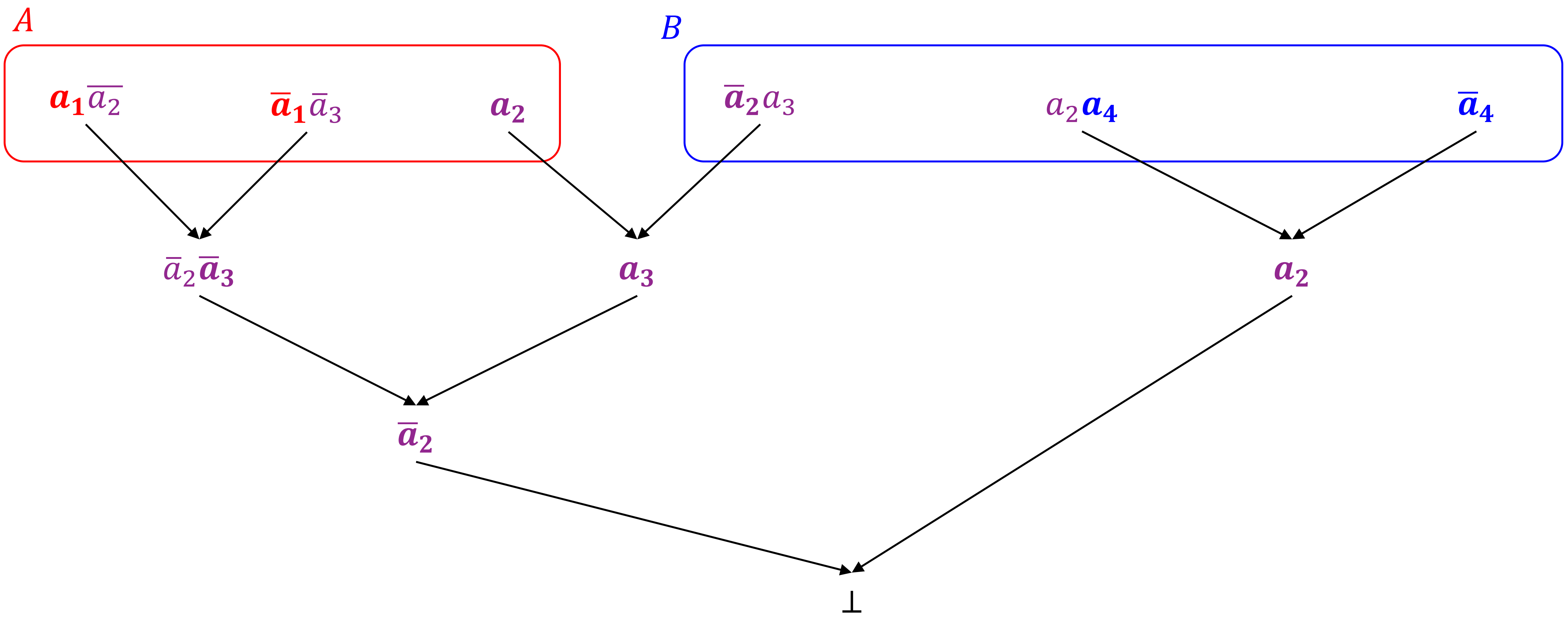

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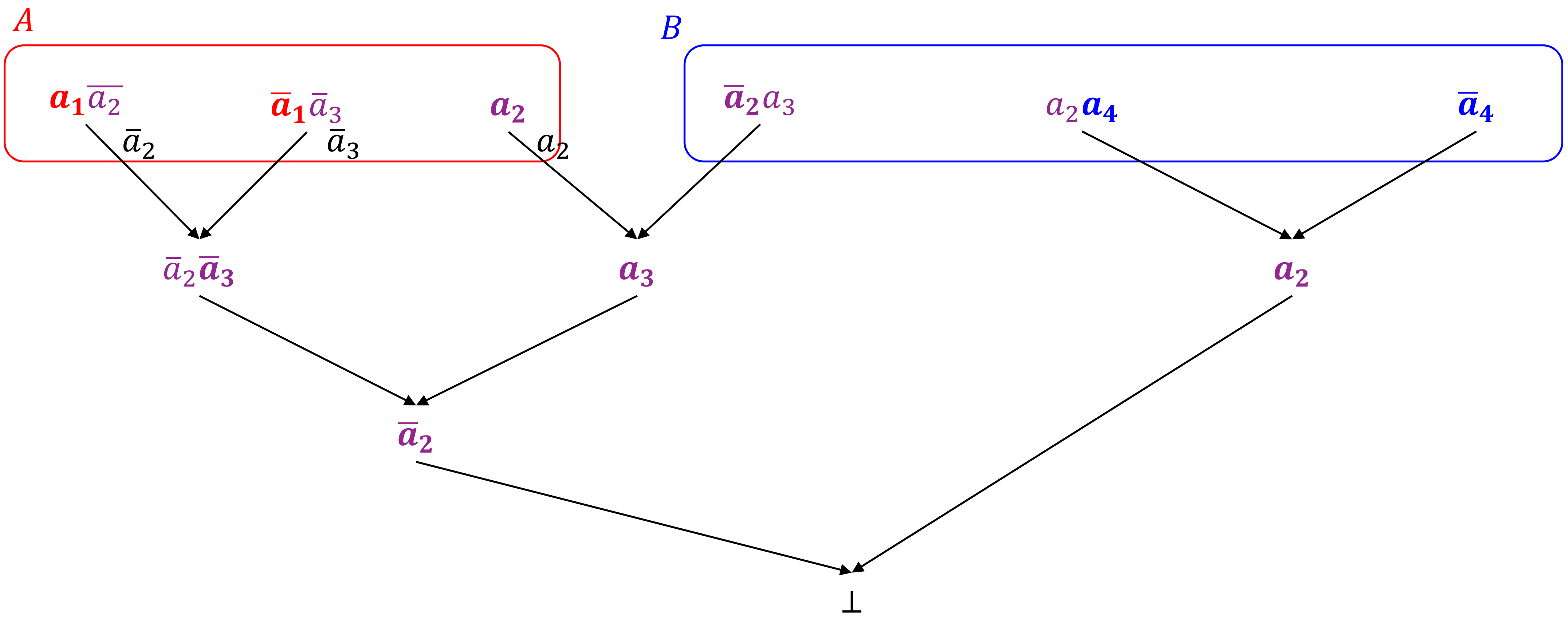

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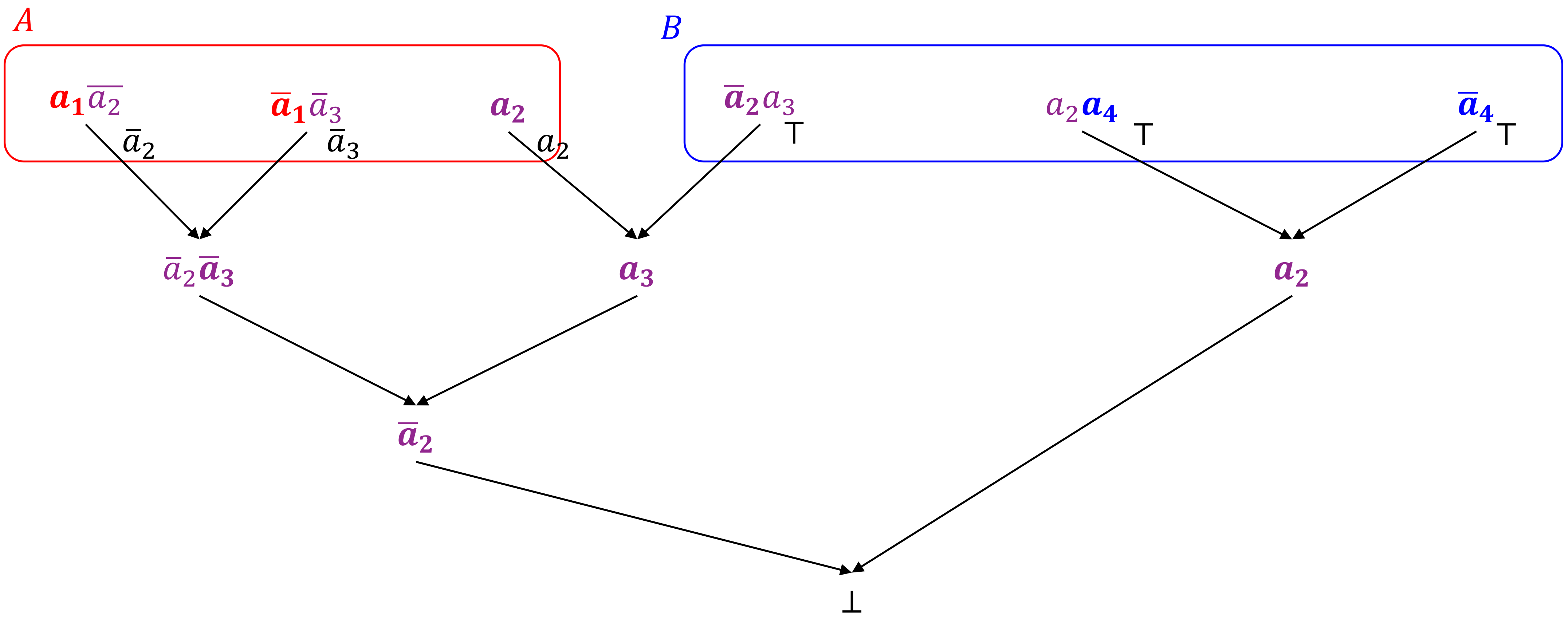

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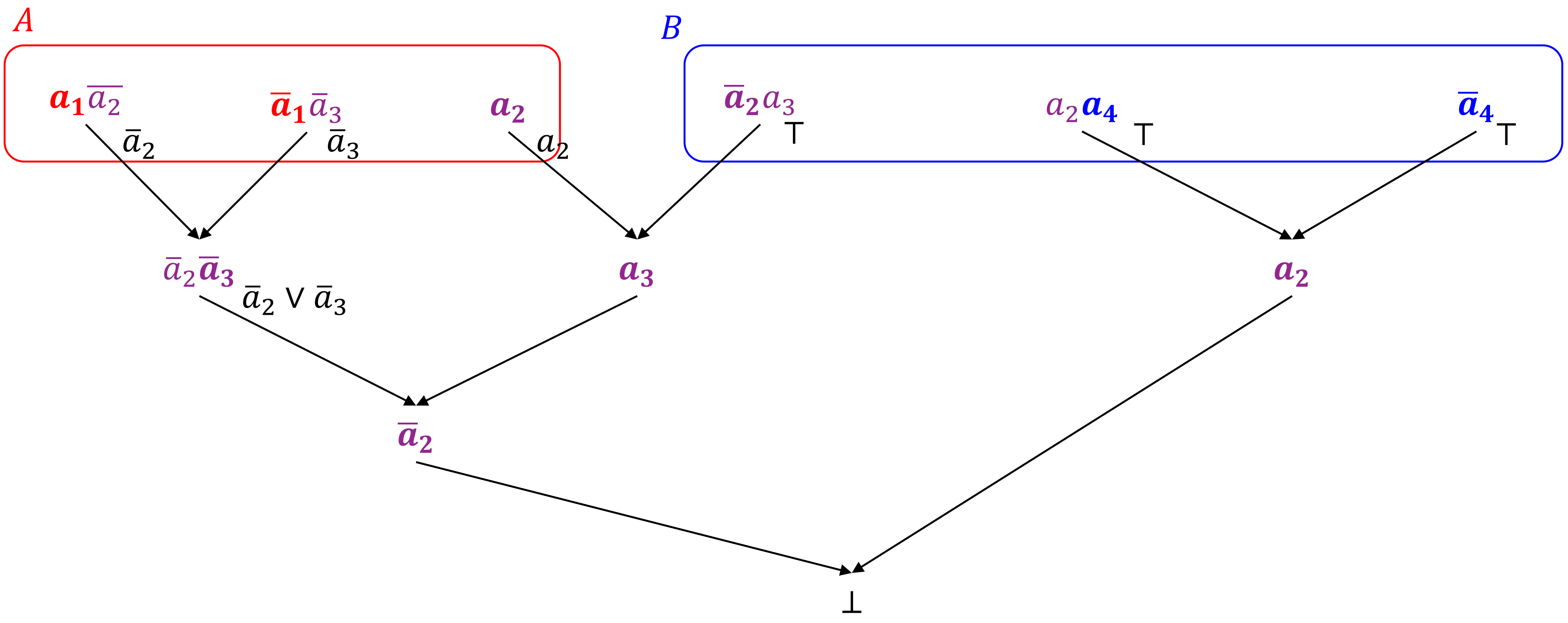

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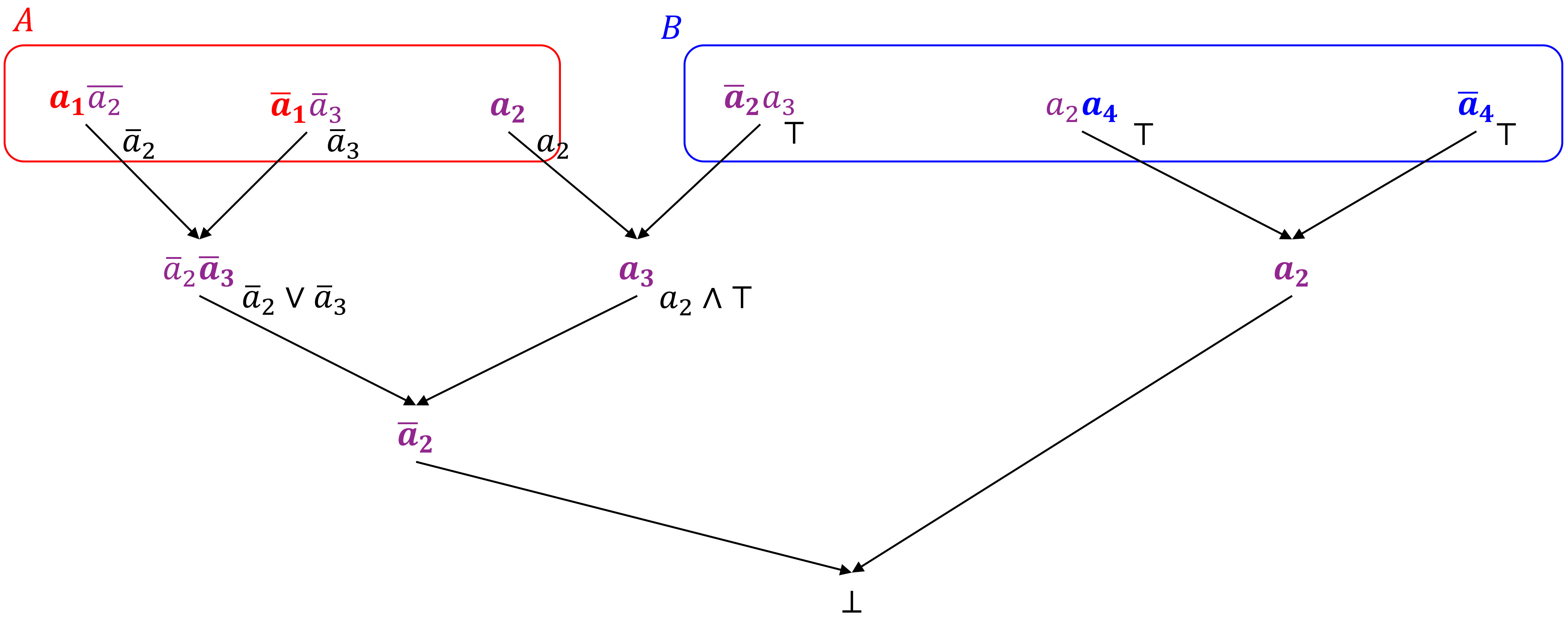

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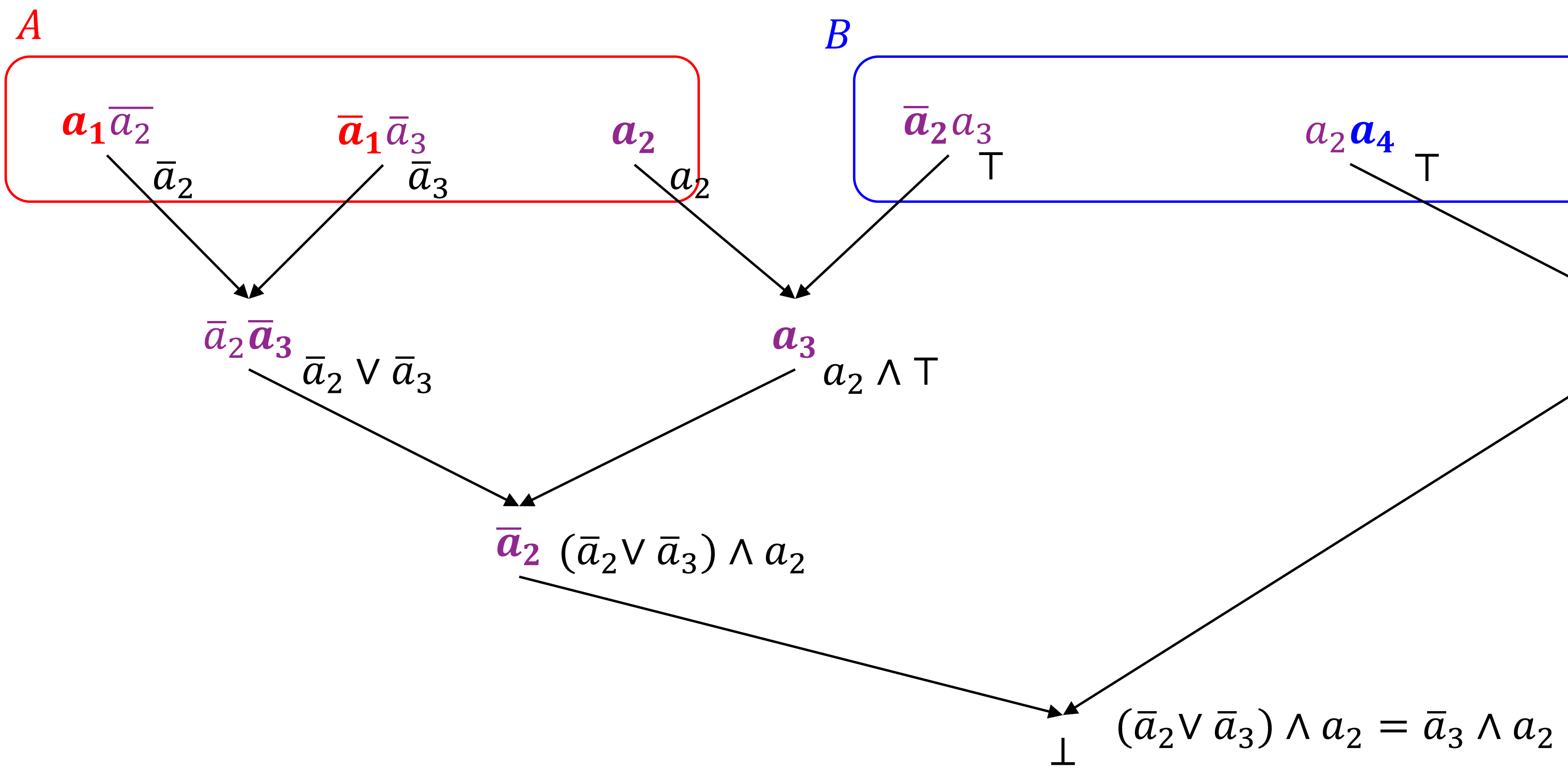


Model Checking with Interpolations

Overapproximation

BMC to prove p:

$$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$$

Suppose we have another transition relation R' with more edges: $R \rightarrow R'$

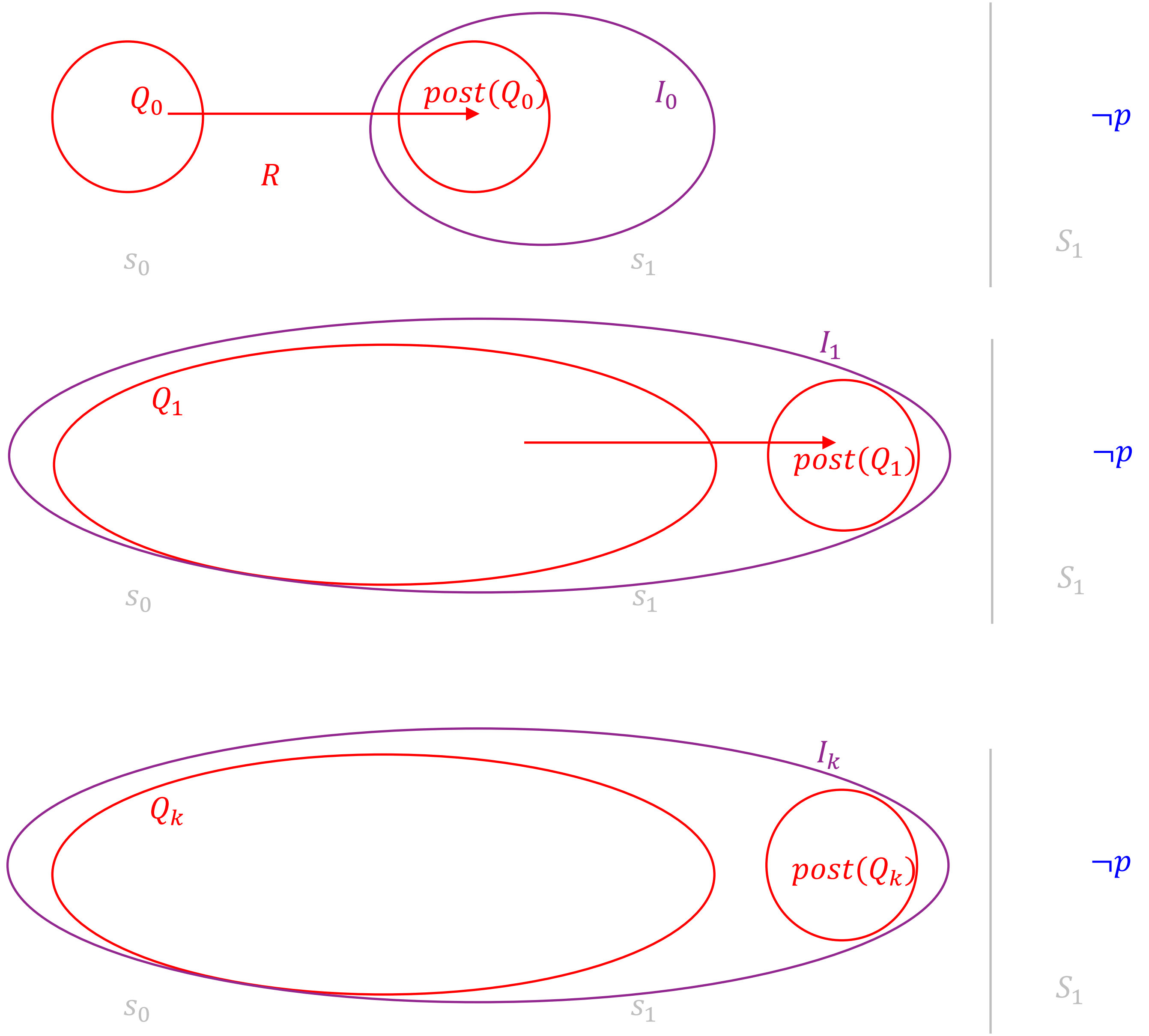
What does this do?

$$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R'(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$$

Idea 1:

if $S_0 \wedge \neg p$ is SAT return "$M \not\models \text{AG}\, p$"
$Q := S_0(s_0)$
while true **do**
  $A := Q(s_0) \wedge R(s_0, s_1)$
  $B := \neg p(s_1)$
  if $A \wedge B$ is SAT **then**
    …
  else
   compute interpolant $I(S_1)$ for A and B;
   if $I(s_0) == Q(s_0)$ then return "$M \models \text{AG}\, p$";
   $Q := Q \vee I(s_0)$;
 end if
end while

$$Q_k \supseteq postimg^k(S_0)$$



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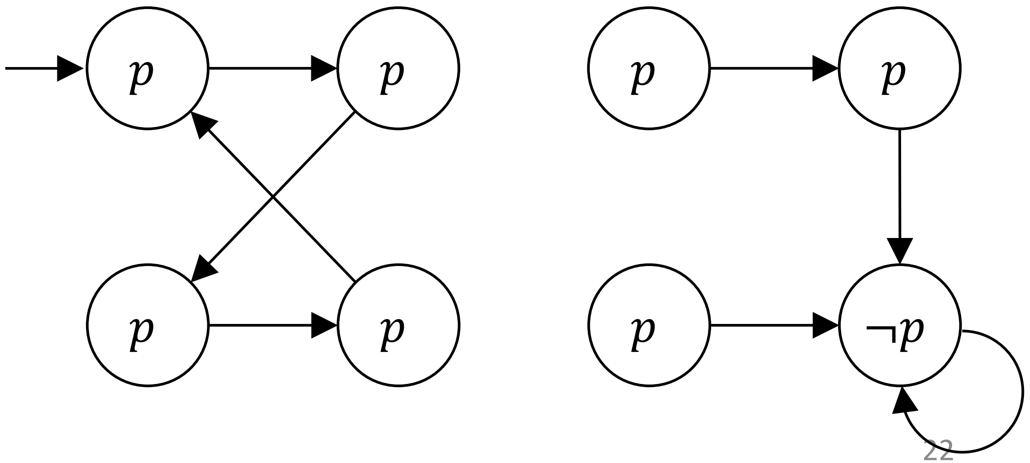
end while

What could go wrong?
And what do we do against it?



First Idea

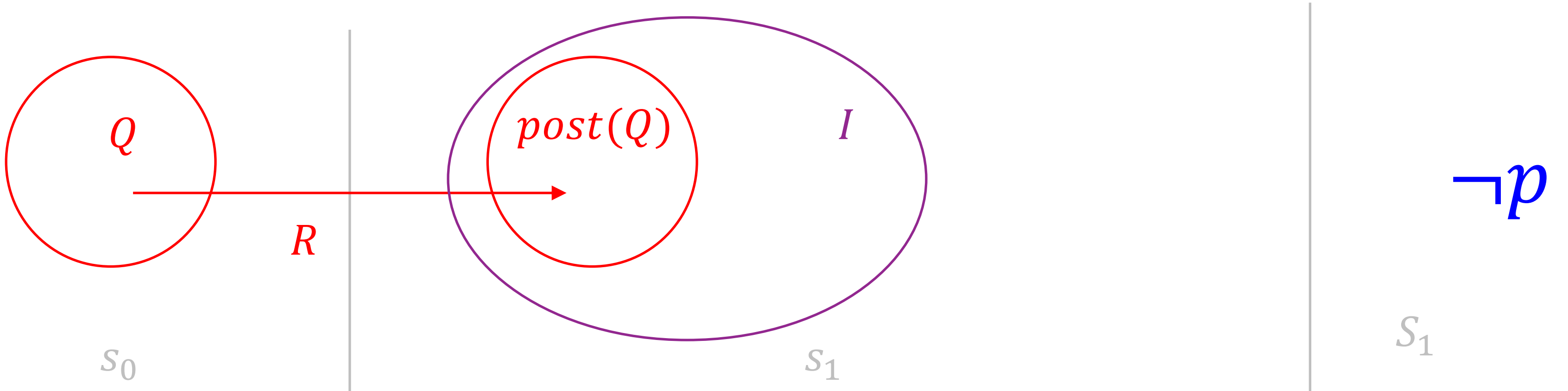
Recall BMC check for $\neg\mathbf{AG}p$:

$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

Instead, start from Q such that $Q \vDash p$

$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \neg p\ (s_1)$.

Suppose ϕ unsatisfiable, $I(s_1)$ is an interpolant



Reachability Checking with Interpolation

Recall BMC check for $\neg\mathbf{AG}p$:
$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

Instead, start from Q such that $Q \vDash p$
$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$.
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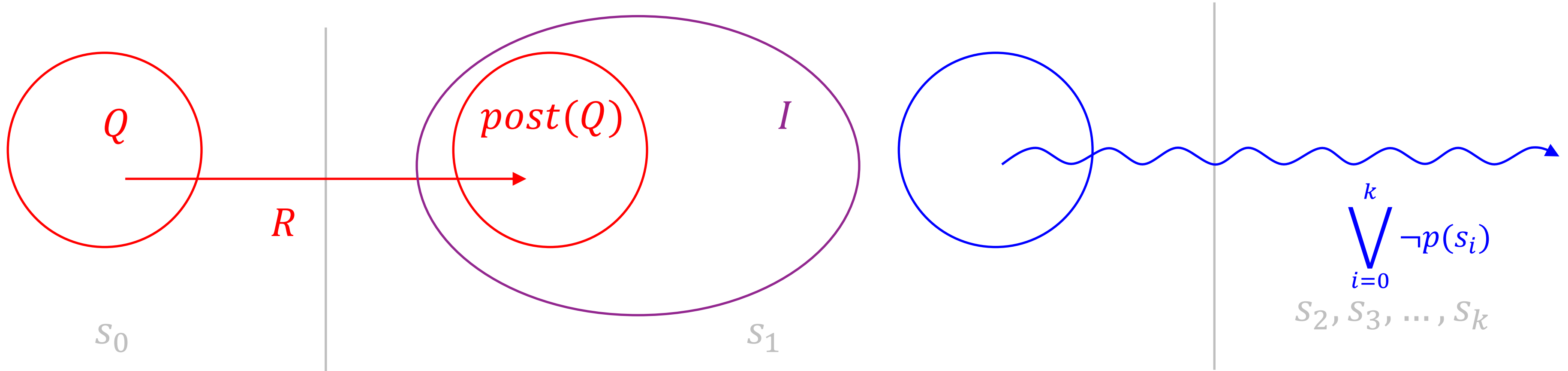


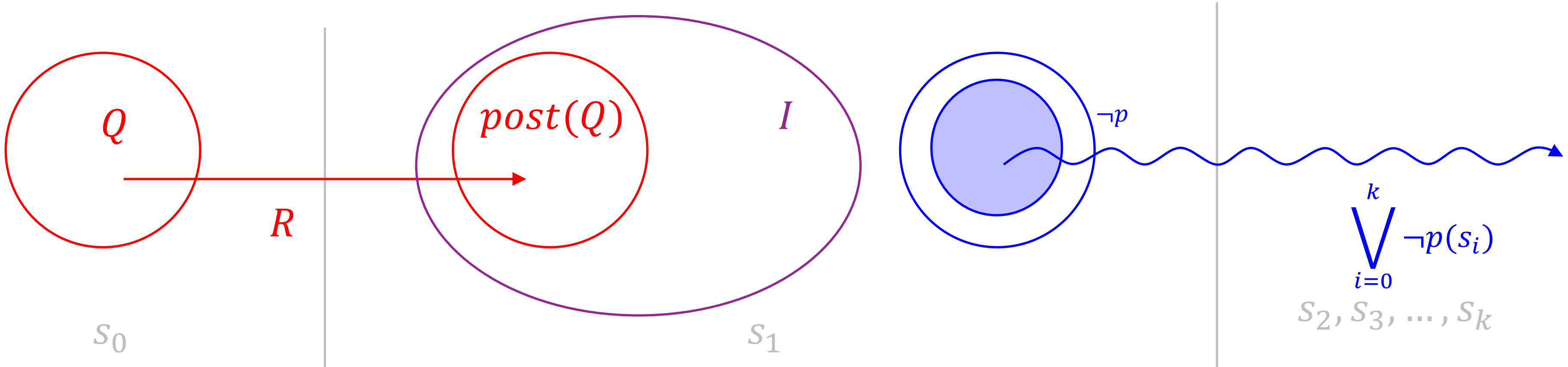
Reachability Checking with Interpolation

Recall BMC check for $\neg \mathbf{AG}p$:
$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

Instead, start from Q such that $Q \vDash p$
$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$.
Suppose ϕ unsatisfiable, $I(s_1)$ is an interpolant

Note 1: $\neg p(s_1) \rightarrow B$
so $I(s_1) \wedge \neg p(s_1) = \bot$

Note 2: $I \supseteq post(Q)$



Interpolant Reachability Idea

$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

1. $Q = S_0$
2. If ϕ not satisfiable, set Q to Q $\cup$ I
3. If Q unchanged, p **is not reachable** *(Interpolants are approximation to post-image),* otherwise goto 2
4. If ϕ is satisfiable and $Q = S_0$, $\neg p$ is **reachable**
5. If ϕ is satisfiable and $Q \neq S_0$, increase k to increase precision of approximation, goto 1.

Procedure terminates when k is diameter of system (or earlier!)

Algorithm

if $S_0 \wedge \neg p$ is SAT return "$M \nvDash \mathrm{AG}\, p$ "
Q := $S_0(s_0)$
while true **do**
  A := $Q(s_0) \wedge R(s_0, s_1)$
  B := $\bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$;
  if $A \wedge B$ is SAT **then**
    if $Q = S_0$ **then return** "$M \nvDash \mathrm{AG}\, p$"; // $\neg p$ can be reached from S_0
    Increase k // Not sure if path to $\neg p$ is real. Increase precision
    Q := $S_0(s_0)$;
  else
    compute interpolant $I(s_1)$ for A and B;
    if $I(s_0) == Q(s_0)$ then return "$M \vDash \mathrm{AG}\, p$";
    Q := $Q \vee I(s_0)$;
  end if
end while

10.4.4 Correctness

If CraigReachability returns "$M \vDash AG\ p$" then $M \vDash AG\ p$

Let Q_i denote Q at iteration i. For all i, $Q_i \leftarrow postimage^i(Q_0)$. If $I \rightarrow Q_i$, we have reached a fixed point $Q^* = Q_i$ so $Q^* \leftarrow postimage^*(Q_0)$. Now because $Q_i \wedge \neg p = \bot$, we have $postimage^*(Q_0) \wedge \neg p = \bot$.

If CraigReachability returns "$M \nvDash AG\ p$" then $M \nvDash AG\ p$

$A \wedge B$ encodes a path from Q_0 to $\neg p$.

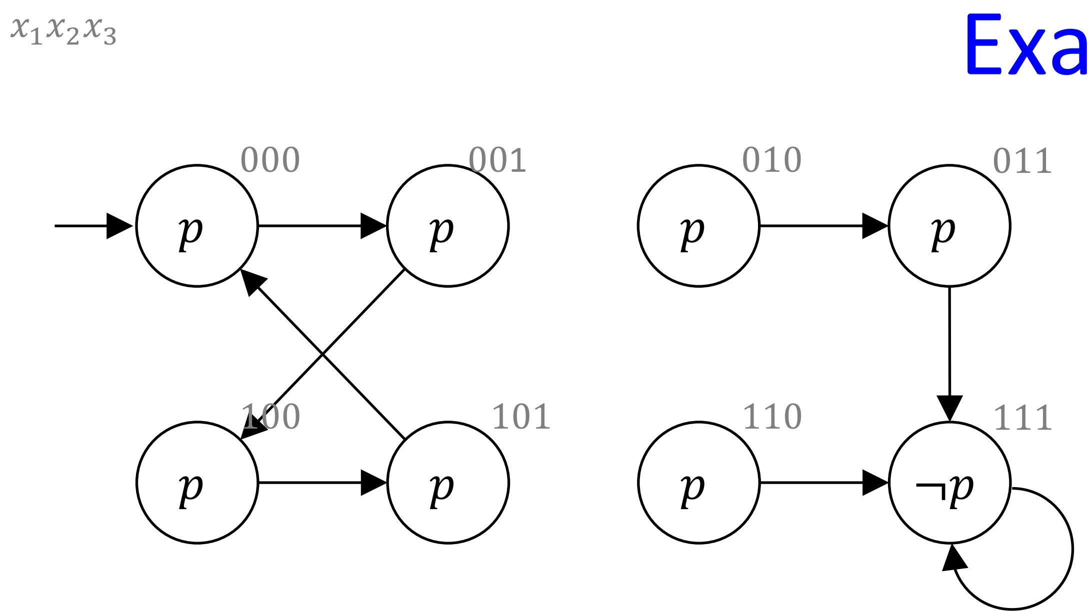
CraigReachability terminates

Note that k increases.

If $M \nvDash AG\ p$, there is a path of length l to $\neg p$ and we will find it when $l = k$.

Suppose $M \vDash AG\ p$. If k is the diameter of the graph, no I and thus no Q_i can contain a state that reaches $\neg p$. Thus, $A \wedge B$ is never SAT and the algorithm terminates because the Q_i cannot grow forever.

Example $\mathbf{AG}\ \boldsymbol{p}$



$$\phi = Q(s_0) \land R(s_0, s_1)$$
$$\land \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \land \bigvee_{i=1}^{k} \neg p(s_i).$$

if $A \land B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

 $Q := S_0(s_0)$;

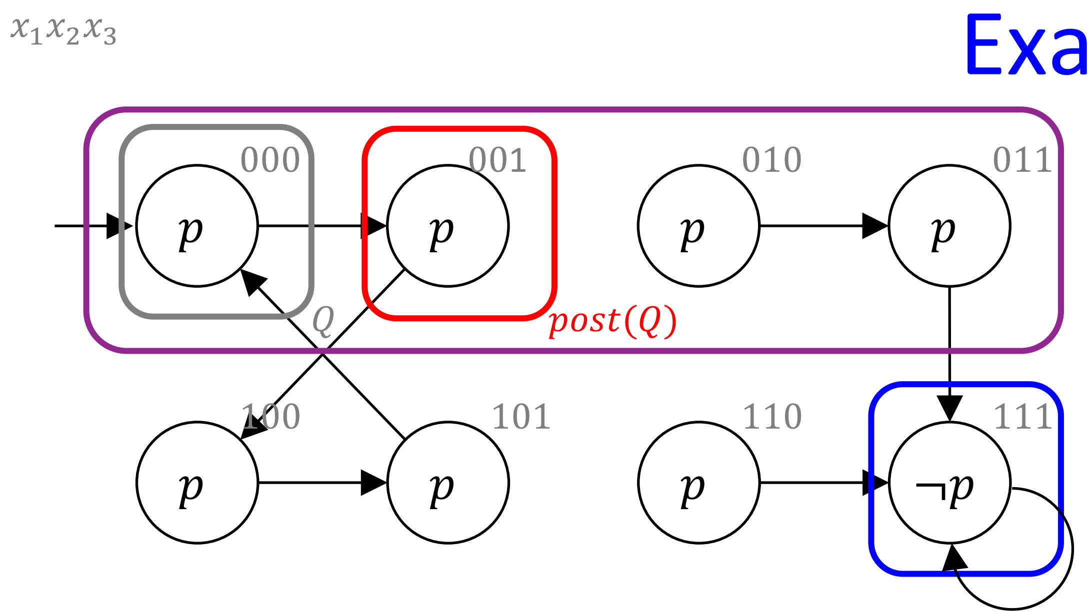
else

  I = interpolate I A and B;

  if $I(s_0) \rightarrow Q$ then return "$M \models \mathrm{AG}\ p$";

 $Q := Q \lor I(s_0)$;

Example AG p



$\phi = Q(s_0) \wedge R(s_0, s_1)$
$\wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$.

$\boldsymbol{k}$ **=1.**

$Q = \neg x_1 \wedge \neg x_2 \wedge \neg x_3 = \{000\}$.

ϕ is UNSAT

Invariant checks first bit: $I = \neg x_1$

if $A \wedge B$ is SAT **then**

if $Q = S_0$ then return "$M \nvDash$ AG p";

increase k

$Q := S_0(s_0)$;

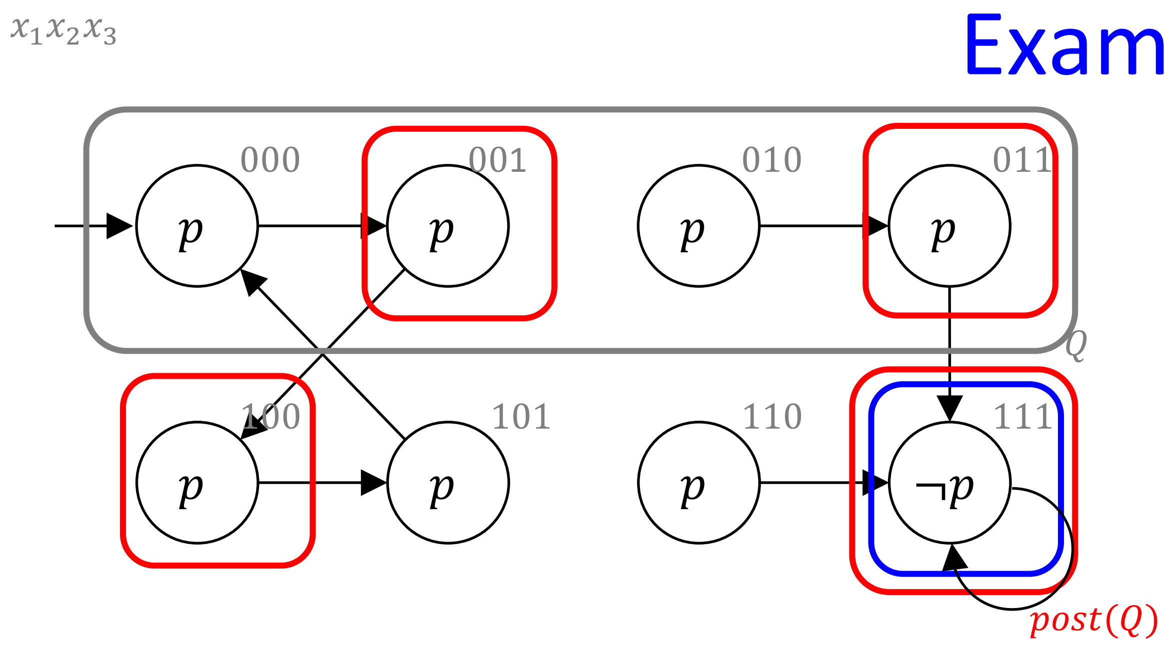
else

compute interpolant I for A and B;

if $I(s_0) \rightarrow Q$ then return "$M \vDash$ AG p";

$Q := Q \vee I(s_0)$;

Example $\mathbf{AG}\ \boldsymbol{p}$



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$\boldsymbol{k}$ **=1.**

$Q = \neg x_1 = \{000,001,010,011\}.$

ϕ is SAT

if $A \wedge B$ is SAT **then**

if $Q = S_0$ then return "$M \nvDash \mathrm{AG}\ p$";

increase k

$Q := S_0(s_0)$;

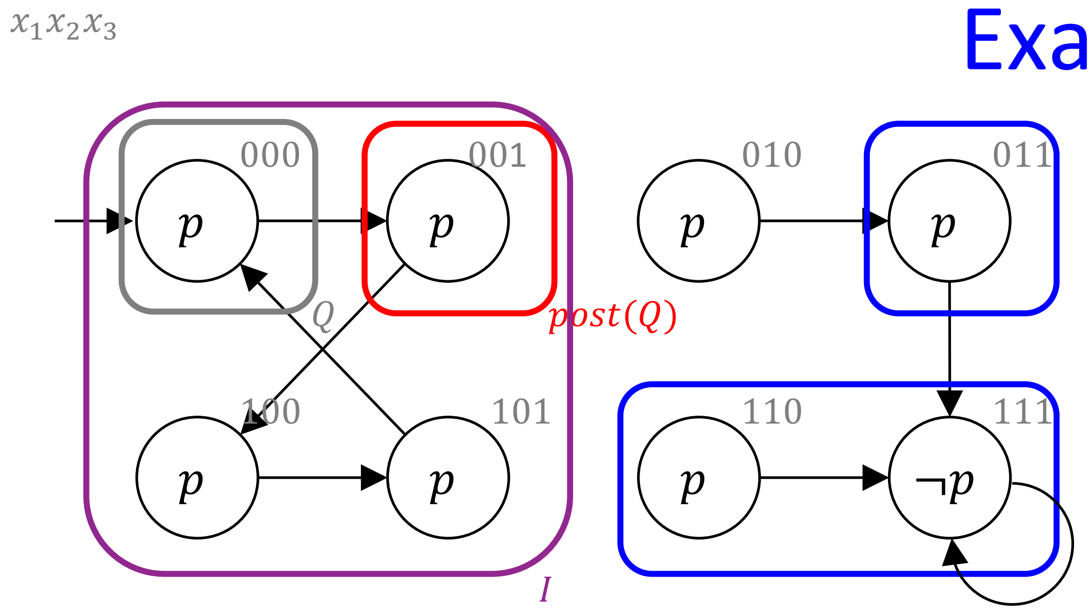
else

I = interpolate I A and B;

if $I(s_0) \rightarrow Q$ then return "$M \vDash \mathrm{AG}\ p$";

$Q := Q \vee I(s_0)$;

Example $\mathbf{AG}\ \boldsymbol{p}$



$\phi = Q(s_0) \wedge R(s_0, s_1)$
$\wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i).$

$\boldsymbol{k = 2.}$

$Q = \neg x_1 \wedge \neg x_2 \wedge \neg x_3 = \{000\}.$

ϕ is UNSAT

Invariant checks 2nd bit: $I = \neg x_2$

if $A \wedge B$ is SAT **then**

if $Q = S_0$ then return "$M \nvDash \mathrm{AG}\ p$";

increase k

$Q := S_0(s_0)$;

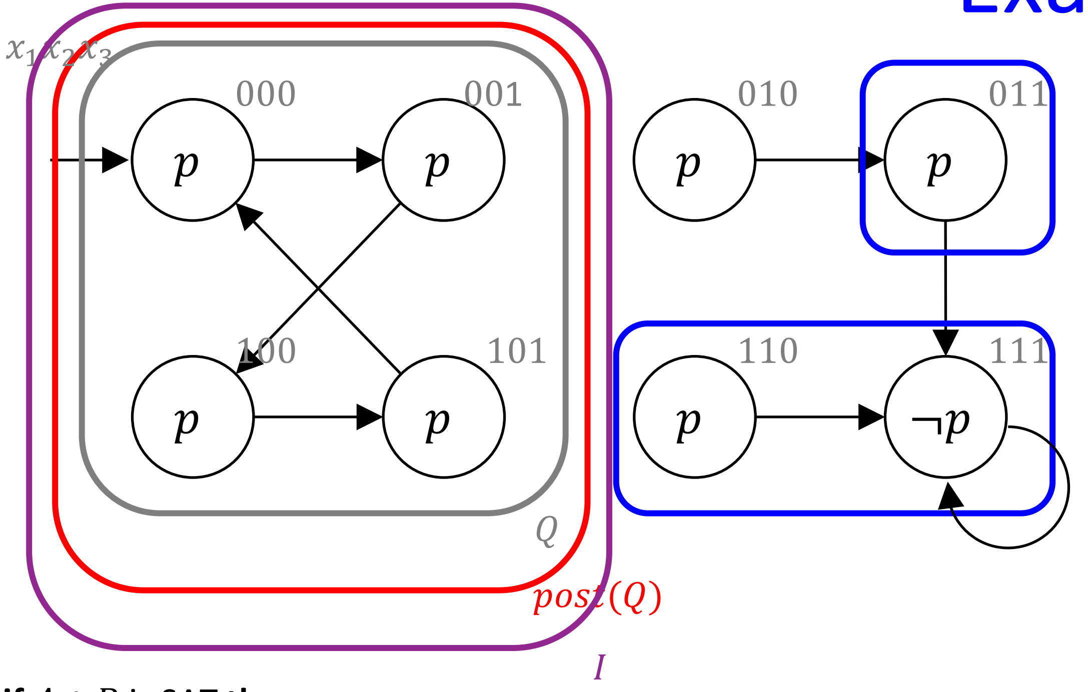
else

I = interpolate I A and B;

if $I(s_0) \rightarrow Q$ then return "$M \vDash \mathrm{AG}\ p$";

$Q := Q \vee I(s_0)$;

Example AG p



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if $Q = S_0$ then return "$M \not\models$ AG p";

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I = interpolate I A and B;

if $I(s_0) \rightarrow Q$ then return "$M \models$ AG p";

$Q := Q \vee I(s_0)$;

$$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i).$$

$\boldsymbol{k = 2.}$

$$Q = \neg x_2 = \{000, 001, 100, 101\}$$

ϕ is UNSAT

$\boldsymbol{I = \neg x_2 = Q.}$

Algorithm terminates.

How Did I Pick the Interpolants?

What I did

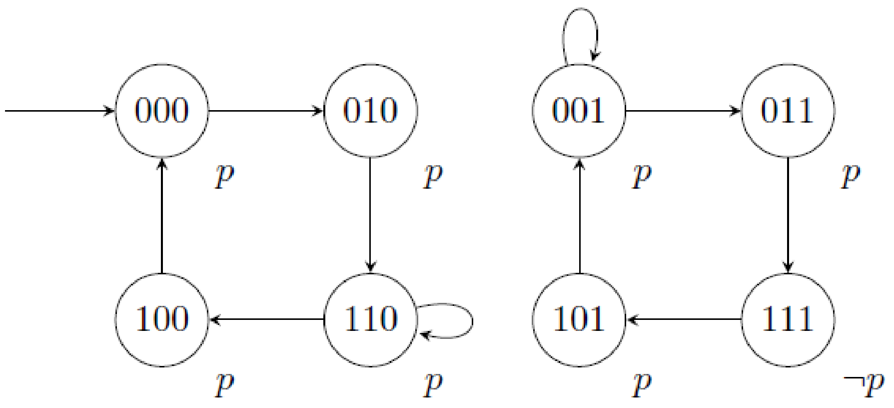
- Start with $\mathrm{A} = \ postimg(Q)$
- Perform each of the following steps
  1. Can I throw away x_3? (Is $(\exists x_3. A) \cap B\ = \emptyset$?) If yes, $A\ \coloneqq \exists x_3. A$
  2. Can I throw away x_2? If yes, $A\ \coloneqq \exists x_2. A$
  3. Can I throw away x_1? If yes, $A\ \coloneqq \exists x_1. A$

This hack only works because the postimg(Q) is a state or a cube!

In the homework, you will get CNFs. Try getting rid of clauses.

Note that A is always a valid interpolant.

Consider the following Kripke structure K, with states $(x_1, x_2, x_3) \in \{0,1\}^3$ and atomic proposition p.



Task 1. [**50 points**] We want to use k-induction to prove that p is always true.

1.1 Will k-induction succeed in proving the property? If so, what is the smallest k such that k-induction proves the property to be true? [**10 point**]

1.2 Write the k induction formulae, both base case and induction case, for $k = 2$. [**20 points**]

1.3 Are the formulae satisfiable? Explain. [**20 points**]

For task 1.2, you can use the formulas R, S_0, and p for the transition relation, the initial states, and the property p, respectively, without explicitly stating the concrete expression of the formulas.

Task 2. [**50 points**] Use Model Checking with Craig Interpolants to prove that p is always true.

Clearly indicate the steps. Show the interpolants as formulas, for anything else, you can use set notation. You can also draw the sets, but use enough copies of the Kripke structure to make sure we can understand your steps, at least one for every k.

Use the same heuristic shown in class to find the interpolants. The heuristic shown in class is a hack, but it works in this example.