Lecture Notes for

# Logic and Computability

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# Natural Deduction for Predicate Logic

In this chapter, we will discuss the natural deduction calculus for predicate logic. We will extend the set of rules we have discussed for propositional logic by adding new rules for quantifiers. As in the natural deduction calculus for propositional logic, we will discuss *introduction* and *elimination* rules for the *quantifiers* and the *equality* predicate.

## 7.1 Natural Deduction Rules

### The ∀-Elimination Rule

We start by discussing the rule for eliminating the universal quantifier ∀:

$$\frac{\forall x\ \varphi}{\varphi\left[t/x\right]}\ \forall_e$$

The rule states that if $\forall x\ \varphi$ is true, we are allowed to replace the $x$ in $\varphi$ with any term $t$, under the condition that $t$ has to be free for $x$ in $\varphi$, and conclude that $\varphi\left[t/x\right]$ is also true. Recall that $\varphi\left[t/x\right]$ is obtained by replacing all free occurrences of $x$ in $\varphi$ by $t$. Since $\varphi$ is assumed to be true for all $x$, then $\varphi$ should also be true for any term $t$.

**Example 1**

Give the proof for the following sequent:

$$\forall x\ \big(\neg P(x) \rightarrow Q(x)\big), \neg Q(t)\ \ \vdash\ \ P(t)$$

**Solution.**

| | | |
|---|---|---|
| 1. | $\forall x\ \big(\neg P(x) \rightarrow Q(x)\big)$ | prem. |
| 2. | $\neg Q(t)$ | prem. |
| 3. | $\neg P(t) \rightarrow Q(t)$ | ∀e 1 |
| 4. | $\neg\neg P(t)$ | MT 3,2 |
| 5. | $P(t)$ | ¬¬e 4 |

Note that if you apply the $\forall_e$ rule, you can use for the substitution any term $t$ (free for $x$ in $\varphi$) which is helpful in your current proof.

## The ∀-Introduction Rule

Now, let us take a look at the rule for the introduction of a universal quantifier ∀:

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\varphi\left[x_0/x\right] \end{array}}\quad x_0 \text{ fresh}}{\forall x\ \varphi}\ \forall_i$$

In order to introduce a formula $\varphi$ that is universally quantified, we have to assume that $\varphi$ holds under an arbitrary choice of variable. Therefore, the rule states, that if we are starting with a **fresh variable** $x_0$ and we are able to prove $\varphi[x_0/x]$, we can derive $\forall x\ \varphi$.

As we have seen in the natural deduction calculus for propositional logic, we have to introduce a proof box that defines the scope of the freshly introduced variable. When applying this rule, there are two things to consider about the fresh variable: First, the variable needs to be *fresh*, i.e. it must not appear in a previous part of the proof, and (2) the variable is bound to the scope, meaning that it must not be used outside the box it has been introduced in.

**Example 2**

Give the proof for the following sequent:

$$\forall x\ \big(P(x) \to Q(x)\big), \forall x\ P(x)\ \ \vdash\ \ \forall x\ Q(x).$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\forall x\ \big(P(x) \to Q(x)\big)$ | prem. |
| 2. | | $\forall x\ P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0) \to Q(x_0)$ | $\forall$e 1 |
| 4. | | $P(x_0)$ | $\forall$e 2 |
| 5. | | $Q(x_0)$ | $\to_e$ 3,4 |
| 6. | | $\forall x\ Q(x)$ | $\forall$i 3-5 |

(Lines 3–5 are enclosed in a box.)

The structure of this proof is guided by the fact that the conclusion is a $\forall$ formula, therefore the application of the $\forall i$ rule is needed. So we set up the box controlling the scope of $x_0$, and we need to prove $Q(x_0)$ inside the box in order to be able to conclude $\forall x\ Q(x)$ outside of the box. Using $\forall e$, we get the two instances of the premises $P(x_0)$ and $P(x_0) \to Q(x_0)$ used to prove $Q(x_0)$.

**Example 3**

Give the proof for the following sequent:

$$\forall x \forall y\ P(x, y)\ \ \vdash\ \ \forall a \forall b\ P(a, b)$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\forall x \forall y\ P(x, y)$ | prem. |
| 2. | $x_0$ | $\forall y\ P(x_0, y)$ | $\forall$e 1 |
| 3. | $y_0$ | $P(x_0, y_0)$ | $\forall$e 2 |
| 4. | | $\forall b\ P(x_0, b)$ | $\forall$i 3 |
| 5. | | $\forall a \forall b\ P(a, b)$ | $\forall$i 4 |

(Lines 2–4 are enclosed in a box; line 3 is enclosed in a nested box.)

## The ∃-Introduction Rule

The $\exists_i$ rule is simply:

$$\frac{\varphi\left[t/x\right]}{\exists x\ \varphi}\ \exists_i$$

The rule states, that if $\varphi[t/x]$ is true, we can conclude $\exists x\ \varphi$. This naturally follows, as $\exists x$ only asks for $\varphi$ to be true for some term $t$, dependent on the side condition that $t$ be free for $x$ in $\varphi$.

**Example 4**

Give the proof for the following sequent:

$$\forall x\ \big(P(x) \to Q(x)\big)\ \ \vdash\ \ \exists y\ \big(P(y) \to Q(y)\big)$$

**Solution.**

| | | |
|---|---|---|
| 1. | $\forall x\ \big(P(x) \to Q(x)\big)$ | prem. |
| 2. | $P(t) \to Q(t)$ | ∀e 1 |
| 3. | $\exists y\ \big(P(y) \to Q(y)\big)$ | ∃i 2 |

**Example 5**

Give the proof for the following sequent:

$$\forall x\ \big(P(x) \land Q(x)\big)\ \ \vdash\ \ \exists x\ \big(P(x) \lor Q(x)\big)$$

**Solution.**

| | | |
|---|---|---|
| 1. | $\forall x\ \big(P(x) \land Q(x)\big)$ | prem. |
| 2. | $P(x_0) \land Q(x_0)$ | ∀e 1 |
| 3. | $P(x_0)$ | $\land e_1$ 2 |
| 4. | $P(x_0) \lor Q(x_0)$ | $\lor i_1$ 3 |
| 5. | $\exists x\ \big(P(x) \lor Q(x)\big)$ | ∃i 4 |

## The ∃-Elimination Rule

The rule for eliminating an ∃ relates to the already known $\lor_e$-rule. The $\exists_e$ rule is defined as follows:

$$\dfrac{\exists x\ \varphi \qquad \boxed{\begin{array}{l} x_0 \\ \quad \varphi\,[x_0/x] \text{ ass.} \\ \quad \vdots \\ \quad \chi \end{array}}\ x_0 \text{ fresh}}{\chi}\ \exists_e$$

Just like when eliminating a disjunction, we need to make a case analysis. As $\exists x\ \varphi$ holds, we know that $\varphi$ is true for at least one value of $x$. If we can deduce a formula $\chi$ without the exact knowledge of the value $x_0$, we can deduce that $\chi$ can be deduced simply from the fact that there exists an $x_0$. In order to do so, we construct a case analysis over all possible values by introducing an arbitrary **fresh variable** $x_0$. If by assuming $\varphi[x_0/x]$ we can prove $\chi$ (that does not contain $x_0$), $\chi$ can be deduced outside of the box. Note, that via the box we are introducing two things: (1) the scope of $x_0$, and (2) the scope of the assumption $\varphi[x_0/x]$.

**Example 6**

Give the proof for the following sequent:

$$\forall x\ \big(P(x) \to Q(x)\big), \exists x\ P(x)\ \vdash\ \exists x\ Q(x)$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\forall x\ \big(P(x) \to Q(x)\big)$ | prem. |
| 2. | | $\exists x\ P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0)$ | ass. |
| 4. | | $P(x_0) \to Q(x_0)$ | $\forall$e 1 |
| 5. | | $Q(x_0)$ | $\to$ e 4,3 |
| 6. | | $\exists x\ Q(x)$ | $\exists$i 5 |
| 7. | | $\exists x\ Q(x)$ | $\exists$e 2,3-6 |

(Lines 3–6 are enclosed in a box.)

The motivation for introducing the box in line 3 of this proof is the existential quantifier in the premise $\exists x P(x)$ which has to be eliminated. In line 4 we eliminate the $\forall$ from line 1. Now, we can extract $Q(x_0)$ using line 4 and line 3. In line 6 we introduce an $\exists$ and substitute the $x_0$ again with an $x$.

As the formula in line 6 does not contain $x_0$ any more, we now may close the box in accordance to our $\exists$e rule. To conclude our $\exists$e, which we started with the box at line 3, in line 7 we need to rewrite the same formula as in line 6.

**Example 7**

Consider the following proof and analyse the error made in this proof:

| | | | |
|---|---|---|---|
| 1. | | $\forall x\ \big(P(x) \to Q(x)\big)$ | prem. |
| 2. | | $\exists x\ P(x)$ | prem. |
| 3. | $x_0$ | $P(x_0)$ | ass. |
| 4. | | $P(x_0) \to Q(x_0)$ | $\forall$e 1 |
| 5. | | $Q(x_0)$ | $\to$ e 4,3 |
| 6. | | $Q(x_0)$ | $\exists$e 2,3-5 |
| 7. | | $\exists x\ Q(x)$ | $\exists$i 6 |

**Solution.** Line 6 allows the fresh variable $x_0$ to escape the scope of the box which declares it. This is not allowed. Therefore, the $\exists i$ rule has to be applied already inside of the box like in the proof above.

Boxes may also be nested within each other. But we need to be careful, on where our scopes begin and where they end. To understand the concept of multiple boxes, we take a look at another interesting example.

**Example 8**

Give the proof for the following sequent:

$$\exists x\ P(x), \forall x\ \forall y\ \big(P(x) \to Q(y)\big)\ \vdash\ \forall y\ Q(y)$$

**Solution.**

| | | | | |
|---|---|---|---|---|
| 1. | | | $\exists x\ P(x)$ | prem. |
| 2. | | | $\forall x\ \forall y\ \big(P(x) \to Q(y)\big)$ | prem. |
| 3. | $y_0$ | | | |
| 4. | | $x_0$ | $P(x_0)$ | ass. |
| 5. | | | $\forall y\ \big(P(x_0) \to Q(y)\big)$ | $\forall$e 2 |
| 6. | | | $P(x_0) \to Q(y_0)$ | $\forall$e 5 |
| 7. | | | $Q(y_0)$ | $\to$ e 6,4 |
| 8. | | | $Q(y_0)$ | $\exists$e 1,4-7 |
| 9. | | | $\forall y\ Q(y)$ | $\forall$i 3-8 |

In this example, the first premise is an $\exists$ formula, which requires an $\exists_e$ to be of any use. The conclusion is an $\forall$ formula, which requires the application of the $\forall_i$ rule.

Therefore, this proof has two boxes. The outer box from 3-8 is for introducing $\forall$, whereas the inner box from 4-7 is for eliminating the $\exists$ from line 1. We need to declare for both boxes fresh variables. To keep it simple, we will substitute $y_0$ for $y$ for the outer box and $x_0$ for $x$ for the inner box. Note again, that it is important to not use $x_0$ and $y_0$ outside of their respective boxes.

**Example 9**

Give the proof for the following sequent:

$$\forall x \, \big(P(x) \land Q(x)\big) \;\vdash\; \forall x \, P(x) \land \forall x \, Q(x)\big)$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\forall x \, \big(P(x) \land Q(x)\big)$ | prem. |
| 2. | $x_0$ | $P(x_0) \land Q(x_0)$ | $\forall$e 1 |
| 3. | | $P(x_0)$ | $\land$e$_1$ 2 |
| 4. | | $\forall x \, P(x)$ | $\forall$i 2-3 |
| 5. | $y_0$ | $P(y_0) \land Q(y_0)$ | $\forall$e 1 |
| 6. | | $Q(y_0)$ | $\land$e$_2$ 5 |
| 7. | | $\forall x \, Q(x)$ | $\forall$i 5-6 |
| 8. | | $\forall x \, P(x) \land \forall x \, Q(x)$ | $\land$i 4,7 |

**Example 10**

Give the proof for the following sequent:

$$\exists x \, P(x) \;\vdash\; \neg \forall x \, \neg P(x)$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\exists x \, P(x)$ | prem. |
| 2. | | $\forall x \, \neg P(x)$ | ass. |
| 3. | $x_0$ | $P(x_0)$ | ass. |
| 4. | | $\neg P(x_0)$ | $\forall$ 2 |
| 5. | | $\bot$ | $\neg$e 3,4 |
| 6. | | $\bot$ | $\exists$e 1,4-5 |
| 7. | | $\neg \forall x \, \neg P(x)$ | $\neg$i 2-6 |

**Example 11**

Give the proof for the following sequent:

$$\neg\forall x \left(P(x) \land Q(x) \land R(y)\right) \;\vdash\; \exists x \, \neg\left(P(x) \land Q(x) \land R(y)\right)$$

**Solution.**

| | | |
|---|---|---|
| 1. | $\neg\forall x \left(P(x) \land Q(x) \land R(y)\right)$ | prem. |
| 2. | $P(t) \land Q(t) \land R(y)$ | ass. |
| 3. | $\forall x \left(P(x) \land Q(x) \land R(y)\right)$ | ∀i 2 |
| 4. | $\bot$ | ¬e 1,3 |
| 5. | $\neg\left(P(t) \land Q(t) \land R(y)\right)$ | ¬i 2-4 |
| 6. | $\exists x \, \neg\left(P(x) \land Q(x) \land R(y)\right)$ | ∃i 5 |

**Example 12**

Give the proof for the following sequent:

$$\exists x \, \neg\left(P(x) \land Q(x) \land R(y)\right) \;\vdash\; \neg\forall x \left(P(x) \land Q(x) \land R(y)\right)$$

**Solution.**

| | | | |
|---|---|---|---|
| 1. | | $\exists x \, \neg\left(P(x) \land Q(x) \land R(y)\right)$ | prem. |
| 2. | | $\forall x \left(P(x) \land Q(x) \land R(y)\right)$ | ass. |
| 3. | $t$ | $\neg\left(P(t) \land Q(t) \land R(y)\right)$ | ass. |
| 4. | | $P(t) \land Q(t) \land R(y)$ | ∀e 2 |
| 5. | | $\bot$ | ¬e 3,4 |
| 6. | | $\bot$ | ∃e 1,3-5 |
| 7. | | $\neg\forall x \left(P(x) \land Q(x) \land R(y)\right)$ | ¬i 2-6 |

### 7.1.1 Quantifier Equivalences

A good way to exercise natural deduction proofs, you can consider proving the most commonly used quantifier equivalences. The proofs are interesting, because most of them involve several quantifications over more than just one variable and your proofs will have nested boxes.

Consider the following equivalences and proof their equivalences by proving both directions:

$$\neg\forall x \, \varphi \equiv \exists x \, \neg\varphi$$

$$\neg\exists x\ \varphi \equiv \forall x\ \neg\varphi$$
$$\neg\forall x\ \neg\varphi \equiv \exists x\ \varphi$$
$$\neg\exists x\ \neg\varphi \equiv \forall x\ \varphi$$

**Example 13**

Proof the following quantifier equivalence:

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

**Solution.** For both directions, we create a proof:

$\forall x\ \neg P(x) \ \vdash\ \neg\exists x\ P(x)$

| | | | |
|---|---|---|---|
| 1. | | $\forall x\ \neg P(x)$ | prem. |
| 2. | | $\exists x\ P(x)$ | ass. |
| 3. | $t$ | $P(t)$ | ass. |
| 4. | | $\neg P(t)$ | $\forall$e 1 |
| 5. | | $\bot$ | $\neg$e 3,4 |
| 6. | | $\bot$ | $\exists$e 2,3-5 |
| 7. | | $\neg\exists x\ P(x)$ | $\neg$i 2-6 |

$\neg\exists x\ P(x) \ \vdash\ \forall x\ \neg P(x)$

| | | | |
|---|---|---|---|
| 1. | | $\neg\exists x\ P(x)$ | prem. |
| 2. | $t$ | | |
| 3. | | $P(t)$ | ass. |
| 4. | | $\exists x\ P(x)$ | $\exists$i 3 |
| 5. | | $\bot$ | $\neg$e 1,4 |
| 6. | | $\neg P(t)$ | $\neg$i 3-5 |
| 7. | | $\forall x\ \neg P(x)$ | $\forall$i 2-6 |

### 7.1.2 Counterexamples

If a sequent is not valid, there is no natural deduction proof for such a sequent. In such cases, we construct a counterexample that proofs the sequent to be invalid. As discussed in the chapter about the natural deduction calculus for propositional logic, *we construct a model, that satisfies all the premises but does not satisfy the conclusion.*

**Example 14**

Show that the following sequent is invalid by constructing a counterexample for it:

$$\exists x \left(P(x) \to S\right) \;\vdash\; \exists x\, P(x) \to S$$

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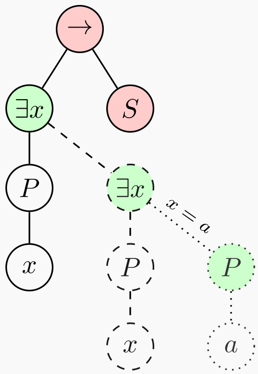
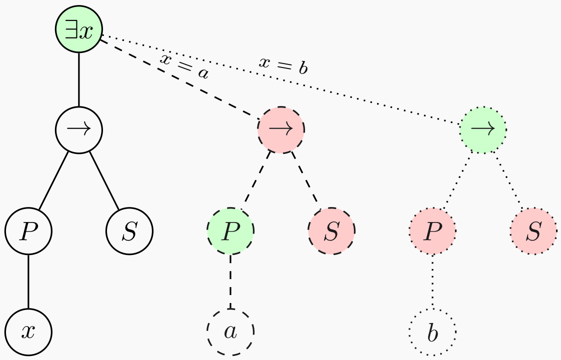
**Solution.** We define the following model $\mathcal{M}$ that serves as a counterexample:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}} = \{a\}$
- $S^{\mathcal{M}} = \bot$

To show, that $\mathcal{M}$ is a counterexample, we first show that $\mathcal{M}$ violates the conclusion, and second that it satisfies the premise.




$\mathcal{M}$ satisfies the premise, but not the conclusion and is therefore a counterexample.

*Showing that* $\mathcal{M} \not\models \exists x\ P(x) \to S$*:* We show that $\mathcal{M}$ does not satisfy the conclusion by drawing a syntax tree. In order for $\mathcal{M}$ to satisfy the $\exists x$ in our formula, there needs to be at least one value for $x$ that $P(x)$ true. When substituting $[a/x]$, we see that $P^{\mathcal{M}}(a) = \mathbf{T}$, which also makes the $\exists x$ node true. The predicate $S$ always evaluates to false. Therefore, the implication results in a $\mathbf{F}$, thus making the conclusion false.

*Showing that* $\mathcal{M} \models \exists x\ \big(P(x) \to S\big)$*:* We again draw the syntax tree to evaluate whether $\mathcal{M}$ satisfies the premise. In order for the $\exists x$ node to become true, we need to fine a value for $x$ that makes the implication node true. Again we first try to substitute $[a/x]$, which results in a true $P(x)$, but in a false implication. If we, however, substitute $[b/x]$, $P(x)$ evaluates to false and thus making the implication true. Thus also our $\exists x$ is true and therefore also the whole premise.

This chapter is based on

.

# List of Definitions