Logic and Computability

Natural Deduction for Predicate Logic

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Motivation

- Extend Natural Deduction to Predicate Logic
  - Richer Language → More powerful proofs
- Basis for “real proofs”

Learning Outcomes

After this lecture…

1. students can explain the predicate-logic specific rules of natural deduction.

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3. for invalid sequents in predicate logic, students can construct counter examples to show that the sequent is invalid.

Learning Outcomes

After this lecture…

1. students can explain the predicate-logic specific rules of natural deduction.
2. for valid sequents in predicate logic, students can construct natural deduction proofs to proof that the sequent is valid.
3. for invalid sequents in predicate logic, students can construct counter examples to show that the sequent is invalid.
4. students can check given natural deduction proofs for correctness.

Plan for Today

- New Rules for Natural Deduction
  - ∀-Quantifier
    - Rules for introduction and elimination
  - ∃-Quantifier
    - Rules for introduction and elimination
- Construct natural deduction proofs
  - Many examples
- Counterexample to proof that sequents are invalid

Proof Rules for Universal Quantification

$$\frac{\forall x \ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall x\ \varphi$ is true, we are allowed to replace the x in φ with any term t.

Substitution $\varphi[t/x]$

Term Variable

- Reads: „φ with t for x„

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- Reads: „φ with t for x„
- Examples:
  - $\varphi = P(f(x, y)) \vee Q(x)$

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  - $\varphi = P\big(f(x, y)\big) \vee Q(x)$
  - $\varphi[a/x] =$

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Term Variable

- Reads: „φ with t for x„
- Examples:
  - $\varphi = P(f(x, y)) \vee Q(x)$
  - $\varphi[a/x] = P(f(a, y)) \vee Q(a)$

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Substitution $\varphi[t/x]$

Conditions for Substitution

Proof Rules for Universal Quantification

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Substitution $\varphi[t/x]$

Conditions for Substitution

- Replace only *free* variables

Proof Rules for Universal Quantification

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall \boldsymbol{x}\ \boldsymbol{\varphi}$ is true, we are allowed to replace the $\boldsymbol{x}$ in $\boldsymbol{\varphi}$ with any term $\boldsymbol{t}$.

Substitution $\boldsymbol{\varphi}[\boldsymbol{t}/\boldsymbol{x}]$

Conditions for Substitution

- Replace only *free* variables
  - $\varphi = \exists y\big(P(x, y) \vee Q(y)\big)$

    bound
  - $\varphi[a/y] =$

Proof Rules for Universal Quantification

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall x\ \varphi$ is true, we are allowed to replace the x in φ with any term t.

Substitution $\varphi[t/x]$

Conditions for Substitution

- Replace only *free* variables
  - $\varphi = \exists y\big(P(x, y) \vee Q(y)\big)$

    bound
  - $\varphi[a/y] = \varphi$

Proof Rules for Universal Quantification

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall x\ \varphi$ is true, we are allowed to replace the x in φ with any term t.

Substitution $\varphi[t/x]$

Conditions for Substitution

- The term t must be *free* for a variable x → No capturing

Proof Rules for Universal Quantification

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall \boldsymbol{x}\ \boldsymbol{\varphi}$ is true, we are allowed to replace the $\boldsymbol{x}$ in $\boldsymbol{\varphi}$ with any term $\boldsymbol{t}$.

Substitution $\boldsymbol{\varphi[t/x]}$

Conditions for Substitution

- The term t must be *free* for a variable x → No capturing
  - $\varphi = \exists x\big(P(x) \vee Q(z)\big)$

    free
  - $\varphi[f(x)/z] =$

Proof Rules for Universal Quantification

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

$\forall \boldsymbol{x}\ \boldsymbol{\varphi}$ is true, we are allowed to replace the $\boldsymbol{x}$ in $\boldsymbol{\varphi}$ with any term $\boldsymbol{t}$.

Substitution $\boldsymbol{\varphi[t/x]}$

Conditions for Substitution

- The term t must be *free* for a variable x → No capturing
  - $\varphi = \exists x\big(P(x) \lor Q(z)\big)$
    - z → free
  - $\varphi[f(x)/z] = \exists x\left(P(x) \lor Q\big(f(x)\big)\right)$
    - x in $f(x)$ → bound

Example 1

- $\forall x\big(\neg P(x) \rightarrow Q(x)\big),\ \neg Q(t) \vdash P(t)$

$$\frac{\forall x\ \varphi}{\varphi\,[t/x]}\ \forall_e$$

Example 1

- $\forall x(\neg P(x) \to Q(x)),\ \neg Q(t) \vdash P(t)$

$$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall_e$$

1.	$\forall x\ (\neg P(x) \to Q(x))$	prem.
2.	$\neg Q(t)$	prem.
3.	$\neg P(t) \to Q(t)$	$\forall$e 1
4.	$\neg\neg P(t)$	MT 3,2
5.	$P(t)$	$\neg\neg$e 4

Example 2

- $\forall x P(x) \wedge \forall x (P(y) \rightarrow Q(x)) \vdash \quad Q(z)$

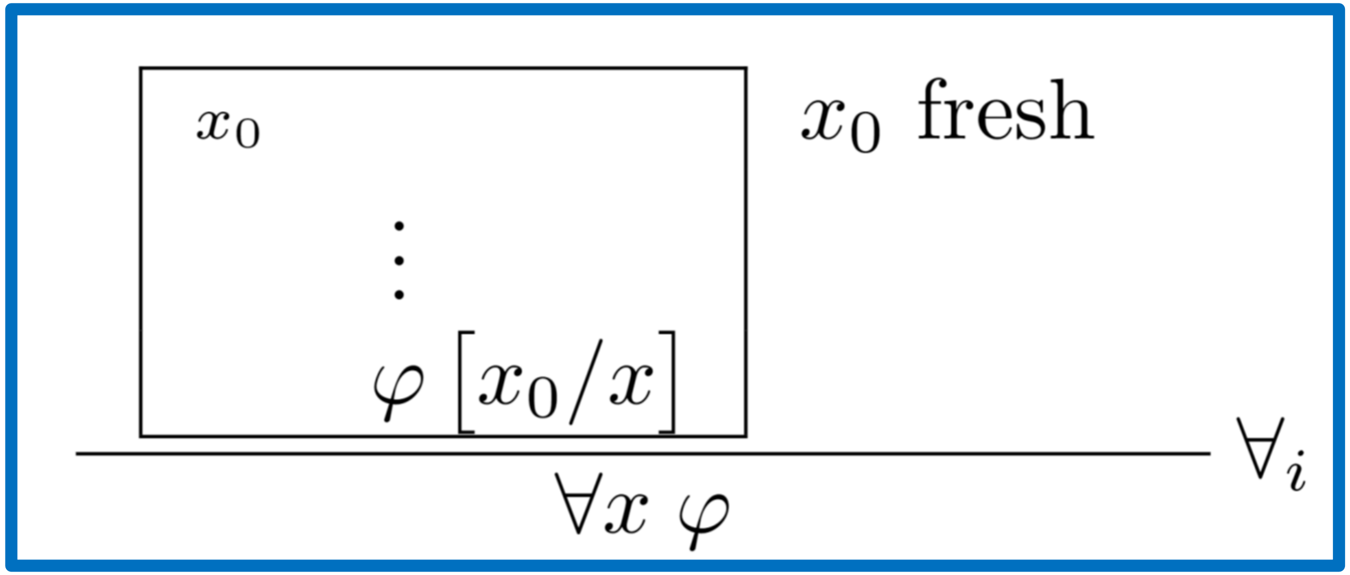
$$\frac{\forall x\, \varphi}{\varphi\,[t/x]}\ \forall_e$$

Example 2

- $\forall x P(x) \land \forall x (P(y) \to Q(x)) \vdash Q(z)$

1.	$\forall x\ P(x) \land \forall x\ (P(y) \to Q(x))$	prem.
2.	$\forall x\ P(x)$	$\land e_1$ 1
3.	$\forall x\ (P(y) \to Q(x))$	$\land e_2$ 1
4.	$P(y)$	$\forall e$ 2
5.	$P(y) \to Q(z)$	$\forall e$ 3
6.	$Q(z)$	$\to e$ 5,4

Proof Rules for Universal Quantification



- If we can proof $\varphi[\boldsymbol{x_0}/x]$ for a **fresh variable $\boldsymbol{x_0}$**, we can derive $\forall \boldsymbol{x}\ \varphi$!

Example 3

- $\forall x\big(P(x) \rightarrow Q(x)\big),\ \forall x P(x) \vdash \quad \forall x Q(x)$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\varphi\left[x_0/x\right] \end{array}}}{\forall x\ \varphi}\ \forall_i \qquad x_0 \text{ fresh}$$

$$\frac{\forall x\ \varphi}{\varphi\left[t/x\right]}\ \forall_e$$

Example 3

- $\forall x(P(x) \rightarrow Q(x)),\ \forall x P(x) \vdash \forall x Q(x)$

1.		$\forall x\ (P(x) \rightarrow Q(x))$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
4.		$P(x_0)$	$\forall$e 2
5.		$Q(x_0)$	$\rightarrow_e$ 3,4
6.		$\forall x\ Q(x)$	$\forall$i 3-5

Example 4

- $\forall x P(x) \vee \forall x\, Q(x) \quad \vdash \quad \forall y(P(y) \vee Q(y))$

Example 4

- $\forall x P(x) \vee \forall x\, Q(x) \ \vdash\ \forall y(\, P(y) \vee Q(y))$

1.		$\forall x\, P(x) \vee \forall x\, Q(x)$	prem.
2.		$\forall x\, P(x)$	ass.
3.	t	$P(t)$	$\forall$e 2
4.		$P(t) \vee Q(t)$	$\vee$i$_1$ 3
5.		$\forall y\, (P(y) \vee Q(y))$	$\forall$i 3-4
6.		$\forall x\, Q(x)$	ass.
7.	s	$Q(s)$	$\forall$e 6
8.		$P(s) \vee Q(s)$	$\vee$i$_2$ 7
9.		$\forall y\, (P(y) \vee Q(y))$	$\forall$i 7-8
10.		$\forall y\, (P(y) \vee Q(y))$	ve 1,2-5,6-9

Proof Rules for Existential Quantification

$$\frac{\varphi\,[t/x]}{\exists x\ \varphi}\ \exists_i$$

- $\exists x$ only asks for φ to be true for some term t
- Side condition: that t be *free* for x in φ

Example 5

- $\forall x\big(P(x) \to Q(y)\big),\ \forall y\big(P(y) \land R(x)\big) \vdash \exists x\, Q(x)$

Example 5

- $\forall x(P(x) \to Q(y)),\ \forall y(P(y) \land R(x)) \vdash \exists x\, Q(x)$

1.	$\forall x\ (P(x) \to Q(y))$	prem.
2.	$\forall y\ (P(y) \land R(x))$	prem.
3.	$P(t) \to Q(y)$	$\forall$e 1
4.	$P(t) \land R(x)$	$\forall$e 2
5.	$P(t)$	$\land e_1$ 4
6.	$Q(y)$	$\to$ e 3
7.	$\exists x\ Q(x)$	$\exists$i 6

Proof Rules for Existential Quantification

$$\frac{\exists x\, \varphi \qquad \boxed{\begin{array}{l} x_0 \\ \qquad \varphi\,[x_0/x] \;\; \text{ass.} \\ \qquad \vdots \\ \qquad \chi \end{array}} \quad x_0 \text{ fresh}}{\chi}\ \exists_e$$

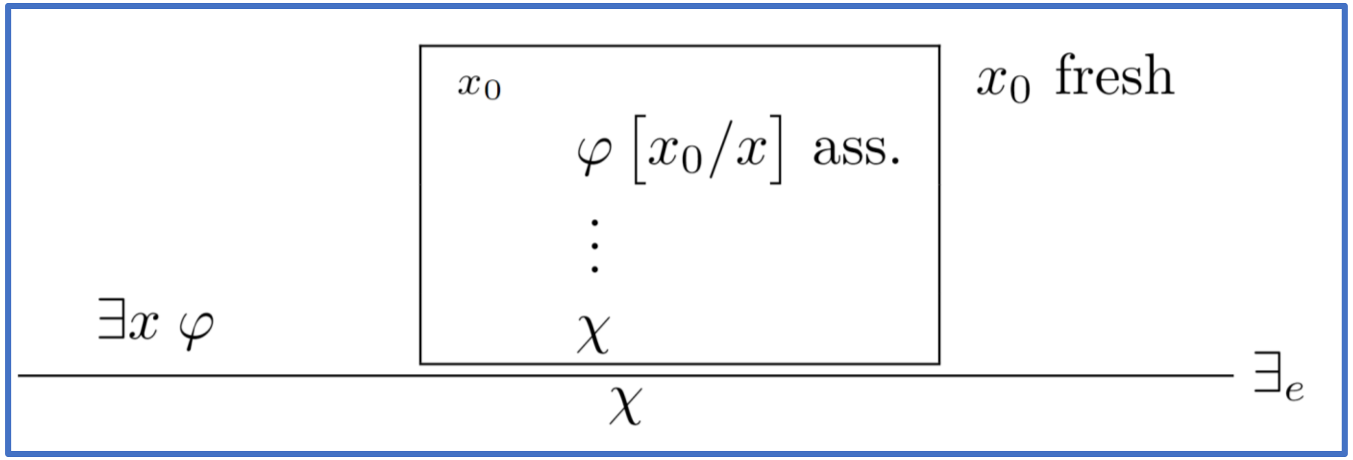
- From $\exists x\ \varphi$, we know that φ is true for at least one value of x

Proof Rules for Existential Quantification

$$\dfrac{\exists x\ \varphi \qquad \boxed{\begin{array}{l} x_0 \\ \qquad \varphi\,[x_0/x] \ \text{ass.} \\ \qquad \vdots \\ \qquad \chi \end{array}} \quad x_0 \ \text{fresh}}{\chi}\ \exists_e$$

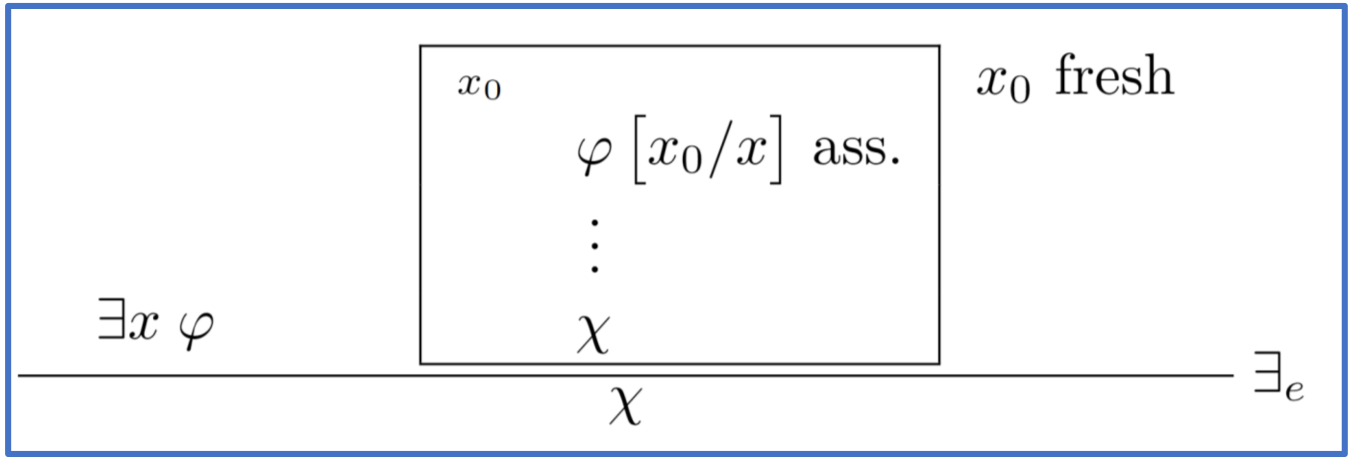
- From $\exists x\ \varphi$, we know that φ is true for at least one value of x
- If we can proof χ without the exact knowledge of the value x_0, then χ can be deduced simply from the fact that there exists an x_0.

Proof Rules for Existential Quantification



- From $\exists x\ \varphi$, we know that φ is true for at least one value of x
- If we can proof χ without the exact knowledge of the value x_0, then χ can be deduced simply from the fact that there exists an x_0.
  - If by assuming $\varphi[x_0/x]$, we can prove χ inside the box, then χ can be deduced outside of the box

Proof Rules for Existential Quantification



- From $\exists x\ \varphi$, we know that φ is true for at least one value of x
- If we can proof χ without the exact knowledge of the value x_0, then χ can be deduced simply from the fact that there exists an x_0.
  - If by assuming $\varphi[x_0/x]$, we can prove χ inside the box, then χ can be deduced outside of the box
- **Important: χ is not allowed to contain x_0!**

Example 6

- $\exists x\big(P(x) \rightarrow Q(y)\big),\ \forall x P(x) \vdash Q(y)$

Example 6

- $\exists x(P(x) \rightarrow Q(y)),\ \forall x P(x) \vdash Q(y)$

1.		$\exists x\ (P(x) \rightarrow Q(y))$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \rightarrow Q(y)$	ass.
4.		$P(x_0)$	$\forall$e 2
5.		$Q(y)$	$\rightarrow$ e 3,4
6.		$Q(y)$	$\exists$e 3-5

Example 7

- $\forall x \neg \big(P(x) \wedge Q(x)\big) \vdash \neg \exists x \, (P(x) \wedge Q(x))$

Example 7

- $\forall x \neg (P(x) \land Q(x)) \vdash \neg \exists x\, (P(x) \land Q(x))$

1.		$\forall x\, \neg(P(x) \land Q(x))$	prem.
2.		$\exists x\, (P(x) \land Q(x))$	ass.
3.	t	$P(t) \land Q(t)$	ass.
4.		$\neg P(t) \land Q(t)$	$\forall$e 1
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 3-5
7.		$\neg\exists x\, (P(x) \land Q(x))$	$\neg$i 2-6

Example 8

- $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\, (\neg P(x) \land \neg Q(x))$

Example 8

- $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\,(\neg P(x) \land \neg Q(x))$

1.		$\exists x\, \neg P(x)$	prem.
2.		$\forall x\, \neg Q(x)$	prem.
3.	x_0	$\neg P(x_0)$	ass.
4.		$\neg Q(x_0)$	$\forall$e 2
5.		$\neg P(x_0) \land \neg Q(x_0)$	$\land$i 3,4
6.		$\exists x\,(\neg P(x) \land \neg Q(x))$	$\exists$i 5
7.		$\exists x\,(\neg P(x) \land \neg Q(x))$	$\exists$e 1, 3-6

Invalid Sequents

$$\exists x\big(P(x) \rightarrow Q(y)\big), \qquad \exists x P(x) \quad \vdash \quad Q(y)$$

Invalid Sequents

$$\exists x\big(P(x) \rightarrow Q(y)\big), \qquad \exists x P(x) \;\vdash\; Q(y)$$

- Model M:
  - $A = \{a, b\}$
  - $P^M = \{a\}$
  - $Q^M = \{a\}$
  - $y \leftarrow b$

Invalid Sequents

$$\exists x\big(P(x) \to Q(y)\big), \qquad \exists x P(x) \quad \vdash \quad Q(y)$$

- Model M:
  - $A \ = \ \{a, b\}$
  - $P^M = \{a\}$
  - $Q^M = \{a\}$
  - $y \leftarrow b$

- $M \vDash \exists x\big(P(x) \to Q(y)\big),\ \exists x\, P(x)$
- $M \nvDash Q(y)$

Invalid Sequents

$$\exists x\big(P(x) \to Q(y)\big), \qquad \exists x P(x) \quad \vdash \quad Q(y)$$

- Model M:
  - $A \;=\; \{a, b\}$
  - $P^M = \{a\}$
  - $Q^M = \{a\}$
  - $y \leftarrow b$

- $M \vDash \exists x\big(P(x) \to Q(y)\big),\; \exists x\, P(x)$
- $M \nvDash Q(y)$

M is a counterexample

Example 9

6.1.33 Consider the following natural deduction proof for the sequent

$$\exists x\, P(x) \vee \exists x\, Q(x) \quad \vdash \quad \exists x\, (P(x) \vee Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.		$\exists x\, P(x) \vee \exists x\, Q(x)$	prem.
2.		$\exists x\, P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \vee Q(x_0)$	$\vee i_1$ 3
5.		$\exists x\, (P(x) \vee Q(x))$	$\exists e$ 2,3-4
6.		$\exists x\, Q(x)$	ass.
7.	x_0	$Q(x_0)$	ass.
8.		$P(x_0) \vee Q(x_0)$	$\vee i_2$ 7
9.		$\exists x\, (P(x) \vee Q(x))$	$\exists e$ 6,7-8
10.		$\exists x\, (P(x) \vee Q(x))$	$\vee e$ 1,2-5,6-9

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1.		$\exists x\, P(x) \vee \exists x\, Q(x)$	prem.
2.		$\exists x\, P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \vee Q(x_0)$	$\vee i_1$ 3
5.		$\exists x\, (P(x) \vee Q(x))$	$\exists$e 2,3-4
6.		$\exists x\, Q(x)$	ass.
7.	x_0	$Q(x_0)$	ass.
8.		$P(x_0) \vee Q(x_0)$	$\vee i_2$ 7
9.		$\exists x\, (P(x) \vee Q(x))$	$\exists$e 6,7-8
10.		$\exists x\, (P(x) \vee Q(x))$	$\vee$e 1,2-5,6-9

$\exists i$ missing (line 4)

$\exists i$ missing (line 8)

Example 9

1.	$\exists x\ P(x) \vee \exists x\ Q(x)$	premise
2.	$\exists x\ P(x)$	assumption
3.	$P(x_0)$	assumption fresh x_0
4.	$P(x_0) \vee Q(x_0)$	$\vee_i 3$
5.	$\exists x\ (P(x) \vee Q(x))$	$\exists_i 4$
6.	$\exists x\ (P(x) \vee Q(x))$	$\exists_e 2, 3-5$
7.	$\exists x\ Q(x)$	assumption
8.	$Q(x_0)$	assumption fresh x_0
9.	$P(x_0) \vee Q(x_0)$	$\vee_i 8$
10.	$\exists x\ (P(x) \vee Q(x))$	$\exists_i 9$
11.	$\exists x\ (P(x) \vee Q(x))$	$\exists_e 7, 8-10$
12.	$\exists x\ (P(x) \vee Q(x))$	$\vee_e 1, 2-6, 7-11$

Example 10

6.1.7 Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \rightarrow Q(x)), \quad \exists x\ P(x) \quad \vdash \quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\forall x\ (P(x) \rightarrow Q(x))$	prem.
2.	$\exists x\ P(x)$	prem.
3.	x_0	
4.	$P(x_0)$	ass.
5.	$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
6.	$Q(x_0)$	$\rightarrow$ e, 4,5
7.	$\forall x\ Q(x)$	$\forall$i 4-6
8.	$\forall x\ Q(x)$	$\exists$e 2,3-7

Example 10

$$\forall x\big(P(x) \to Q(y)\big), \qquad \exists x P(x) \quad \vdash \quad \forall x Q(x)$$

- Model M:
  - $A = \{a, b\}$
  - $P^M = \{a\}$
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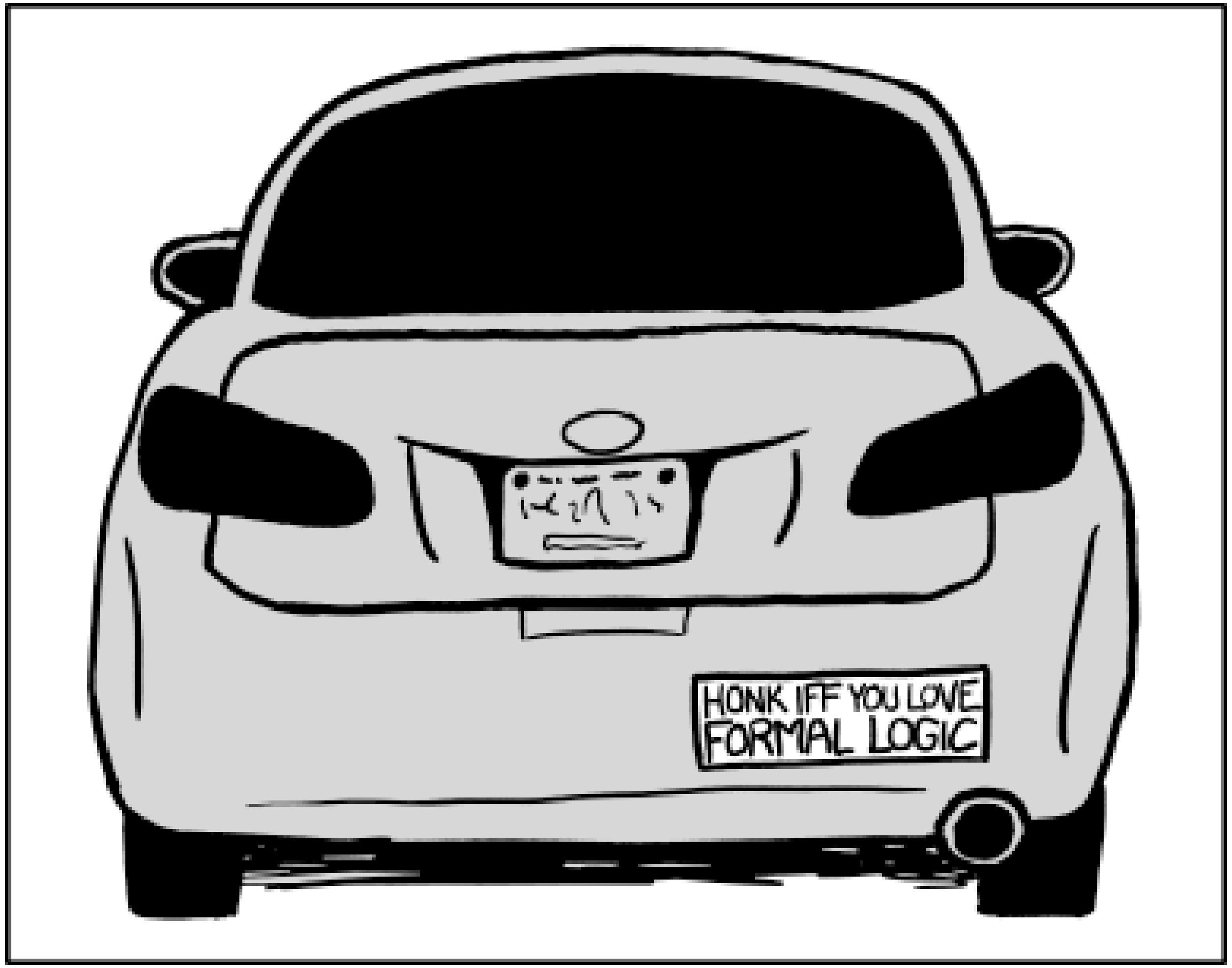
  - $y \leftarrow b$

- $M \vDash \forall x\big(P(x) \to Q(y)\big),\ \exists x\ P(x)$
- $M \nvDash Q(y)$

M is a counterexample

Thank You



https://xkcd.com/1033/