Logic and Computability

Natural Deduction

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https://xkcd.com/1724/

Recap - Topics we discussed so far

- Propositional Logic
  - Syntax and Semantics
- SAT Solving (DPLL)
  - (Efficiently) solve huge formulas
- BDDs
  - Data structure to efficiently store and manipulate formulas

Recap - Topics we discussed so far

- Propositional Logic
  - Syntax and Semantics
- SAT Solving (DPLL)
  - (Efficiently) solve huge formulas
- BDDs
  - Data structure to efficiently store and manipulate formulas
- Today: Proofs
  - Prove that arguments in prop. logic are valid

Motivation – Natural Deduction

- **Example**: Prove that the argumentation is valid

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
2. Alice is not late for her appointment.
3. The plane did arrive late.
4. *Therefore*, there were taxis at the airport.

Motivation – Natural Deduction

- **Example**: Prove that the argumentation is valid

Knowledge that we have.
Facts that we know are true.
→ **Premises**

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
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- **Example**: Prove that the argumentation is valid

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4. *Therefore*, there were taxis at the airport.

Knowledge that we have.
Facts that we know are true.
→ **Premises**

Deduce new knowledge from the sentences 1,2, and 3.
→ **Conclusion**

Motivation – Natural Deduction

- **Example**: Prove that the argumentation is valid

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
2. Alice is not late for her appointment.
3. The plane did arrive late.

---

4. *Therefore*, there were taxis at the airport.

1. $(p \wedge \neg t) \rightarrow l$
2. $\neg l$
3. p

---

4. t

p… the plane arrives late
t … there are taxis at the airport
l… Alice is late for the appointment

Motivation – Natural Deduction

- **Example**: Prove that the argumentation is valid

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
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2. $\neg l$
3. p
4. t

p... the plane arrives late
t ... there are taxis at the airport
l... Alice is late for the appointment

How can we prove that?

- Natural Deduction (TODAY ☺)

Natural Deduction

- Defines set of proof rules
  - Syntactic rewriting rules
  - Apply these rules in succession to infer conclusion from premises

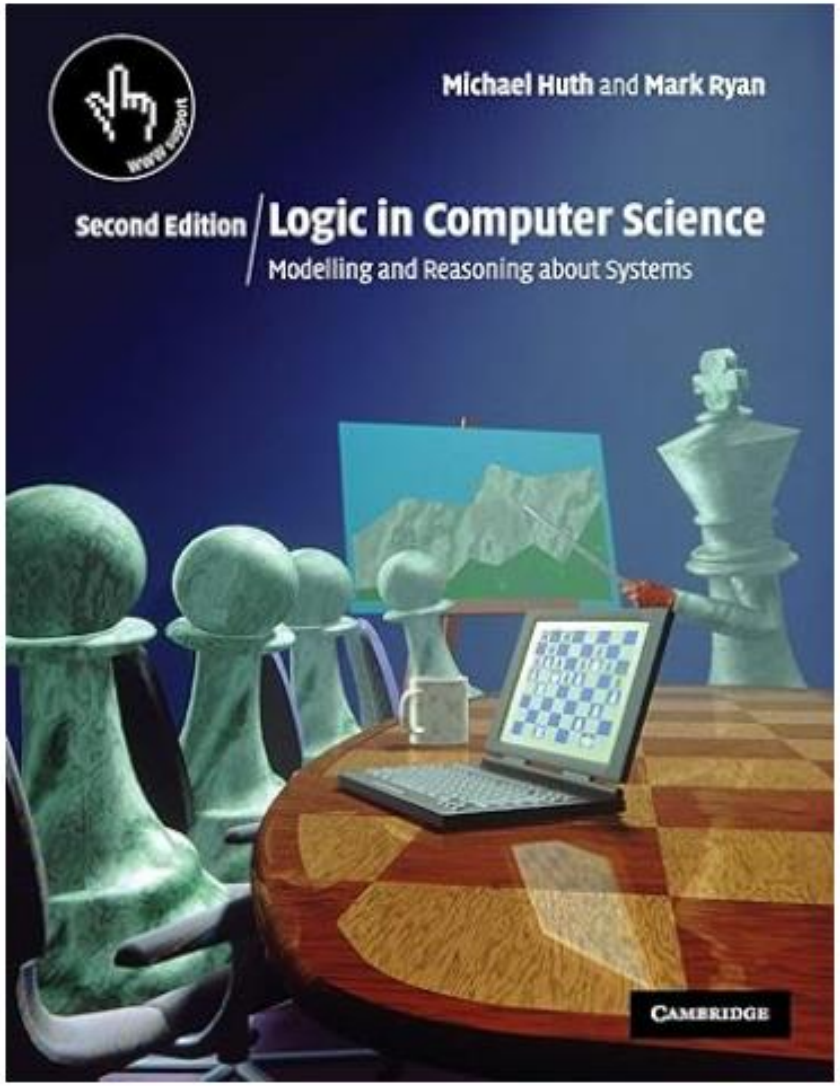
Natural Deduction

- Defines Set of Proof Rules
- Create “watertight” proofs
    - **No** “Dark-Magic” Proofs
    - Proofs can be checked and generated automatically

https://xkcd.com/1724/

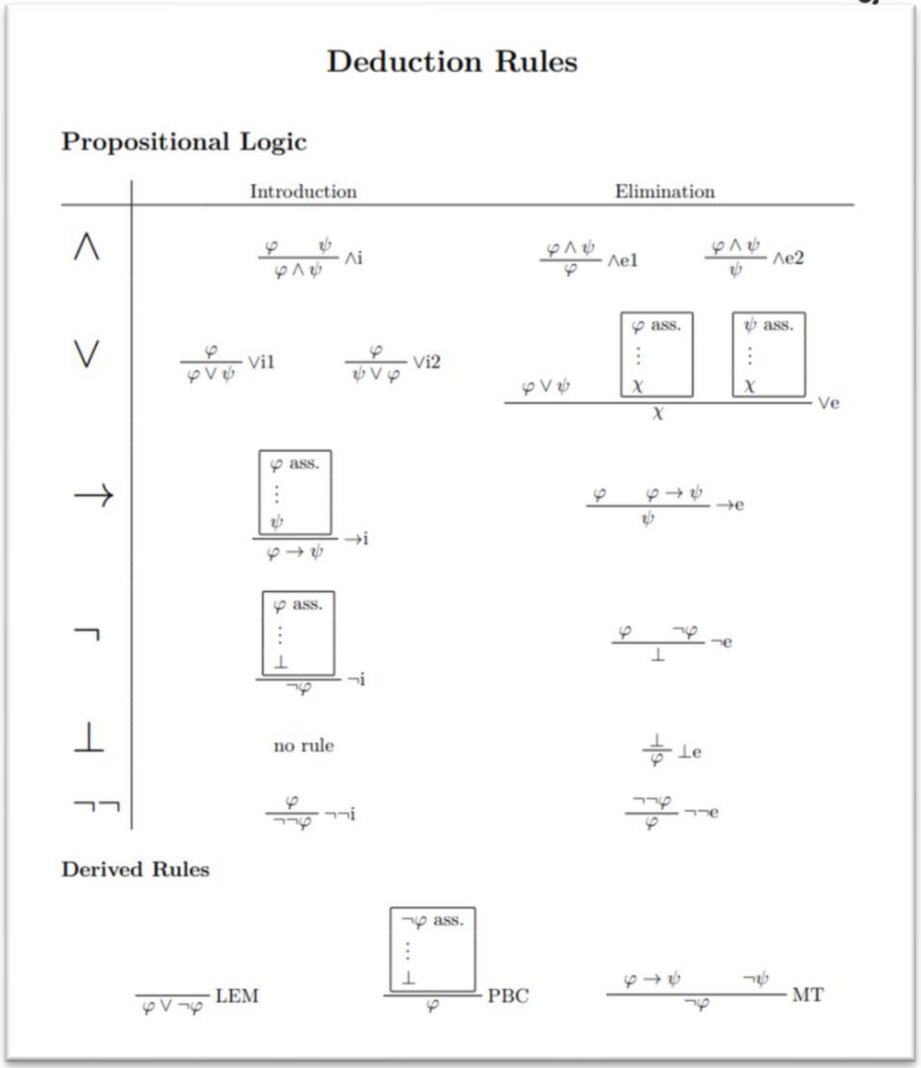
Natural Deduction

- Defines Set of Proof Rules
- Create “watertight” proofs
- Literature:
  - Logic in Computer Science:
    Modelling and Reasoning about Systems
    2nd (Second) edition.
    From M. Huth and M. Ryan
  - Section 1.2 Natural Deduction



Outline

- Proof rules
- Valid argument
  - Prove validity via natural deduction
- Invalid argument (flawed)
  - Prove invalidity via counter example
- Soundness and Completeness



Deduction Rules

Propositional Logic

	Introduction	Elimination
$\wedge$	$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{i}$	$\frac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{e1}$ $\quad \frac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{e2}$
$\vee$	$\frac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{i1}$ $\quad \frac{\varphi}{\psi \vee \varphi}\ \vee\mathrm{i2}$	$\frac{\varphi \vee \psi \quad [\varphi \text{ ass.} \ \vdots\ \chi] \quad [\psi \text{ ass.} \ \vdots\ \chi]}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\frac{[\varphi \text{ ass.} \ \vdots\ \psi]}{\varphi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e}$
$\neg$	$\frac{[\varphi \text{ ass.} \ \vdots\ \bot]}{\neg\varphi}\ \neg\mathrm{i}$	$\frac{\varphi \quad \neg\varphi}{\bot}\ \neg\mathrm{e}$
$\bot$	no rule	$\frac{\bot}{\varphi}\ \bot\mathrm{e}$
$\neg\neg$	$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg\mathrm{i}$	$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg\mathrm{e}$

Derived Rules

$\frac{}{\varphi \vee \neg\varphi}\ \mathrm{LEM}$ $\quad \frac{[\neg\varphi \text{ ass.} \ \vdots\ \bot]}{\varphi}\ \mathrm{PBC}$ $\quad \frac{\varphi \rightarrow \psi \quad \neg\psi}{\neg\varphi}\ \mathrm{MT}$

Learning Outcomes

After this lecture…

1. students can explain the proof rules of ND for prop. logic.

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2. students can construct ND proofs for valid sequents.

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2. students can construct ND proofs for valid sequents.
3. students can construct counterexamples for invalid sequents.

Learning Outcomes

After this lecture…

1. students can explain the proof rules of ND for prop. logic.
2. students can construct ND proofs for valid sequents.
3. students can construct counterexamples for invalid sequents.
4. students can explain (a) what it means that ND for prop. logic is sound and complete and (b) can explain the consequences of it’s soundness and completeness.

Sequents (Arguments)

$$\underbrace{\phi_1,\ \phi_2,\ \dots,\ \phi_n}_{\text{Premises}} \vdash \underbrace{\Psi}_{\text{Conclusion}}$$

Premises **Conclusion**

$\vdash$ … single turnstile

read: „entails“

„proofs“

(Latex: \vdash)

Rules for Conjunction

- AND-Introduction Rule

Premises

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

Conclusion

Rules for Conjunction

- AND-Introduction Rule

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge i$$

- AND-Elimination Rules

$$\frac{\varphi \wedge \psi}{\varphi} \wedge e_1 \qquad \frac{\varphi \wedge \psi}{\psi} \wedge e_2$$

Example: $p, q, r \vdash p \land q \land r$

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

Example: $p, q, r \vdash p \land q \land r$

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

1.	p	premise
2.	q	premise
3.	r	premise
4.	$p \land q$	$\land i$ 1,2
5.	$p \land q \land r$	$\land i$ 4,3

Example: $p \land q, r \vdash q \land r$

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

$$\frac{\varphi \land \psi}{\phi} \land e_1$$

Example: $p \wedge q, r \vdash q \wedge r$

$$\frac{\varphi \wedge \psi}{\phi} \wedge e_1$$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge i$$

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Example: $(p \land q) \land r, s \land t \vdash q \land s$

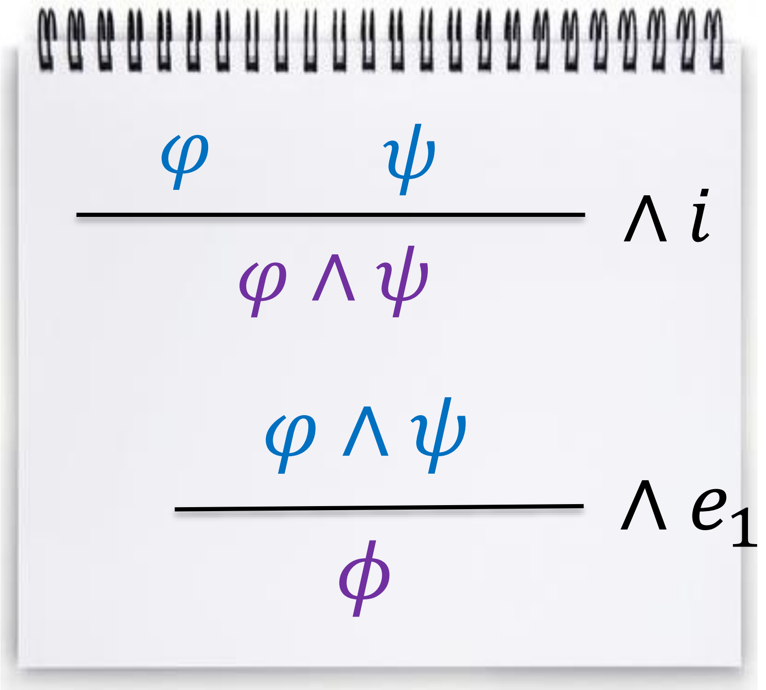
1.	$(p \land q) \land r$	premise
2.	$s \land t$	premise
3.		
4.		
5.		
6.	$q \land s$	

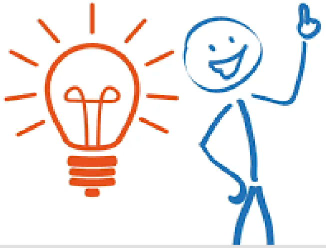
$$\frac{\varphi \quad \psi}{\varphi \land \psi} \land i$$

$$\frac{\varphi \land \psi}{\phi} \land e_1$$

Example: $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

1.	$(p \wedge q) \wedge r$	premise
2.	$s \wedge t$	premise
3.	$p \wedge q$	$\wedge e_1$ 1
4.	q	$\wedge e_2$ 3
5.	s	$\wedge e_1$ 2
6.	$q \wedge s$	$\wedge i$ 4,5





Rules for Double Negation

Elimination

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg\mathrm{e}$$

Introduction

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg i$$

Example: $p \land q, \neg q \land r \vdash \neg\neg p \land \neg\neg r$

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg e$$

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg i$$

1.	$p \land q$	prem.
2.	$\neg q \land r$	prem.
	$\neg\neg p \land \neg\neg r$	

Example: $p \land q, \neg q \land r \vdash \neg\neg p \land \neg\neg r$

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg\text{e}$$

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg\text{i}$$

1.	$p \land q$	prem.
2.	$\neg q \land r$	prem.
3.	p	$\land$e1 1
4.	r	$\land$e2 2
5.	$\neg\neg p$	$\neg\neg$i 3
6.	$\neg\neg r$	$\neg\neg$i 4
7.	$\neg\neg p \land \neg\neg r$	$\land$i 5,6

Rules for Implication - Elimination

Elimination

Modus Ponens

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \rightarrow e$$

Derived Elimination Rule -

Modus Tollens

$$\frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi} \; MT$$

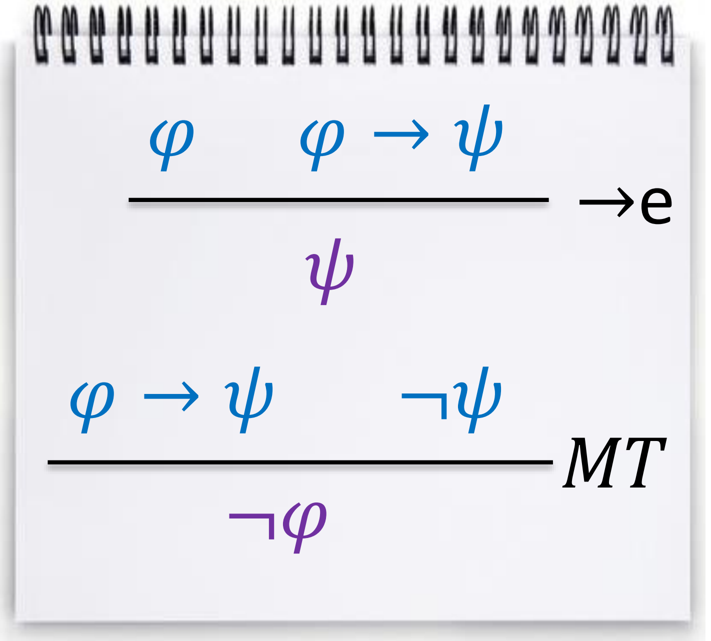
Example: $p, p \to q, p \to (q \to r) \vdash r$

1.	p	prem.
2.	$p \to q$	prem.
3.	$p \to (q \to r)$	prem.
4.		
5.		
6.	r	

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} MT$$

Example: $p, p \to q, p \to (q \to r) \vdash r$



1.	p	prem.
2.	$p \to q$	prem.
3.	$p \to (q \to r)$	prem.
4.	$q \to r$	$\to$e 1,3
5.	q	$\to$e 1,2
6.	r	$\to$e 4,5

Example: $\neg p \to (q \to r), \neg p, \neg r \vdash \neg q$

1.	$\neg p \to (q \to r)$	prem.
2.	$\neg p$	prem.
3.	$\neg r$	prem.
4.		
5.	$\neg q$	

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} MT$$

Example: $\neg p \to (q \to r), \neg p, \neg r \vdash \neg q$

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to e$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} MT$$

1.	$\neg p \to (q \to r)$	prem.
2.	$\neg p$	prem.
3.	$\neg r$	prem.
4.	$q \to r$	$\to$e 1,2
5.	$\neg q$	MT 4,3

Rules for Implication

Introduction

Temporary assumption φ

Box: Scope of assumption φ

φ assum.

...

ψ

$$\frac{\boxed{\begin{array}{l}\varphi \quad \text{assum.} \\ \dots \\ \psi\end{array}}}{\varphi \rightarrow \psi} \rightarrow \text{i}$$

Hint

- If conclusion is of the form $\varphi \rightarrow \psi$, apply $\rightarrow i$ immediately

Example: $p \to q, q \to r \vdash \ p \to r$

1.	$p \to q$	prem.
2.	$q \to r$	prem.
3.		
4.		
5.		
6.	$p \to r$	

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to e$$

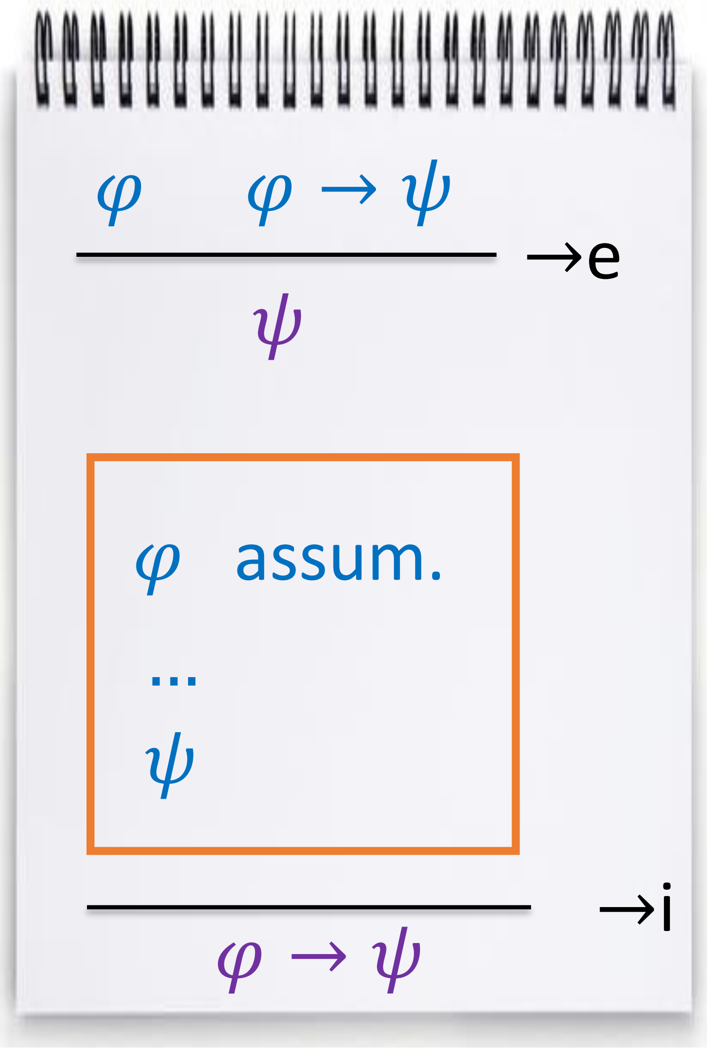
φ	assum.
...	
ψ	

$$\to i$$

$$\varphi \to \psi$$

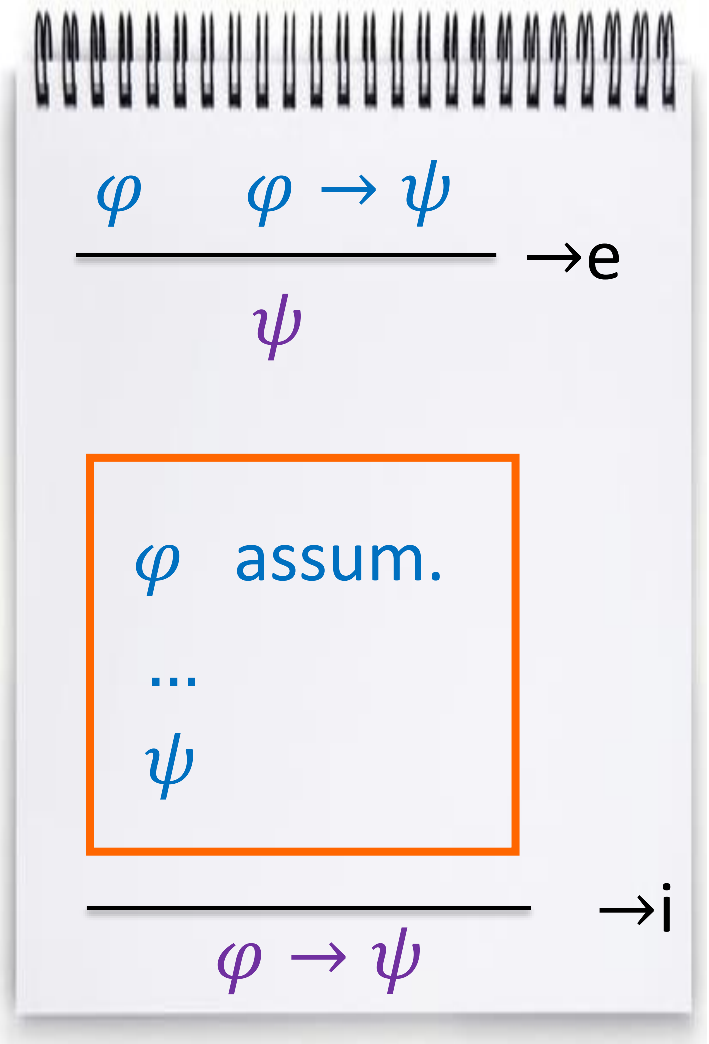
Example: $p \to q, q \to r \vdash p \to r$

1.	$p \to q$	prem.
2.	$q \to r$	prem.
3.	p	ass.
4.	q	$\to$e 3,1
5.	r	$\to$e 2,4
6.	$p \to r$	$\to$i 3 – 5



Example: $p \to (q \land r), (q \to s) \vdash p \to (s \land r)$

1.	$p \to (q \land r)$	prem.
2.	$q \to s$	prem.
3.		
4.		
5.		
6.		
7.		
8.		
9.	$p \to (s \land r)$	



Example: $p \to (q \wedge r), (q \to s) \vdash p \to (s \wedge r)$

1.	$p \to (q \wedge r)$	prem.
2.	$q \to s$	prem.
3.	p	ass.
4.	$q \wedge r$	$\to$e 1,3
5.	q	$\wedge$e1 4
6.	s	$\to$e 2,5
7.	r	$\wedge$e2 4
8.	$s \wedge r$	$\wedge$i 6,7
9.	$p \to (s \wedge r)$	$\to$i 3-8

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \to\text{e}$$

φ assum.

...

ψ

$$\overline{\varphi \to \psi} \to\text{i}$$

Rules for Disjunction

Introduction

$$\frac{\varphi}{\varphi \vee \psi} \vee i_1$$

$$\frac{\varphi}{\psi \vee \varphi} \vee i_2$$

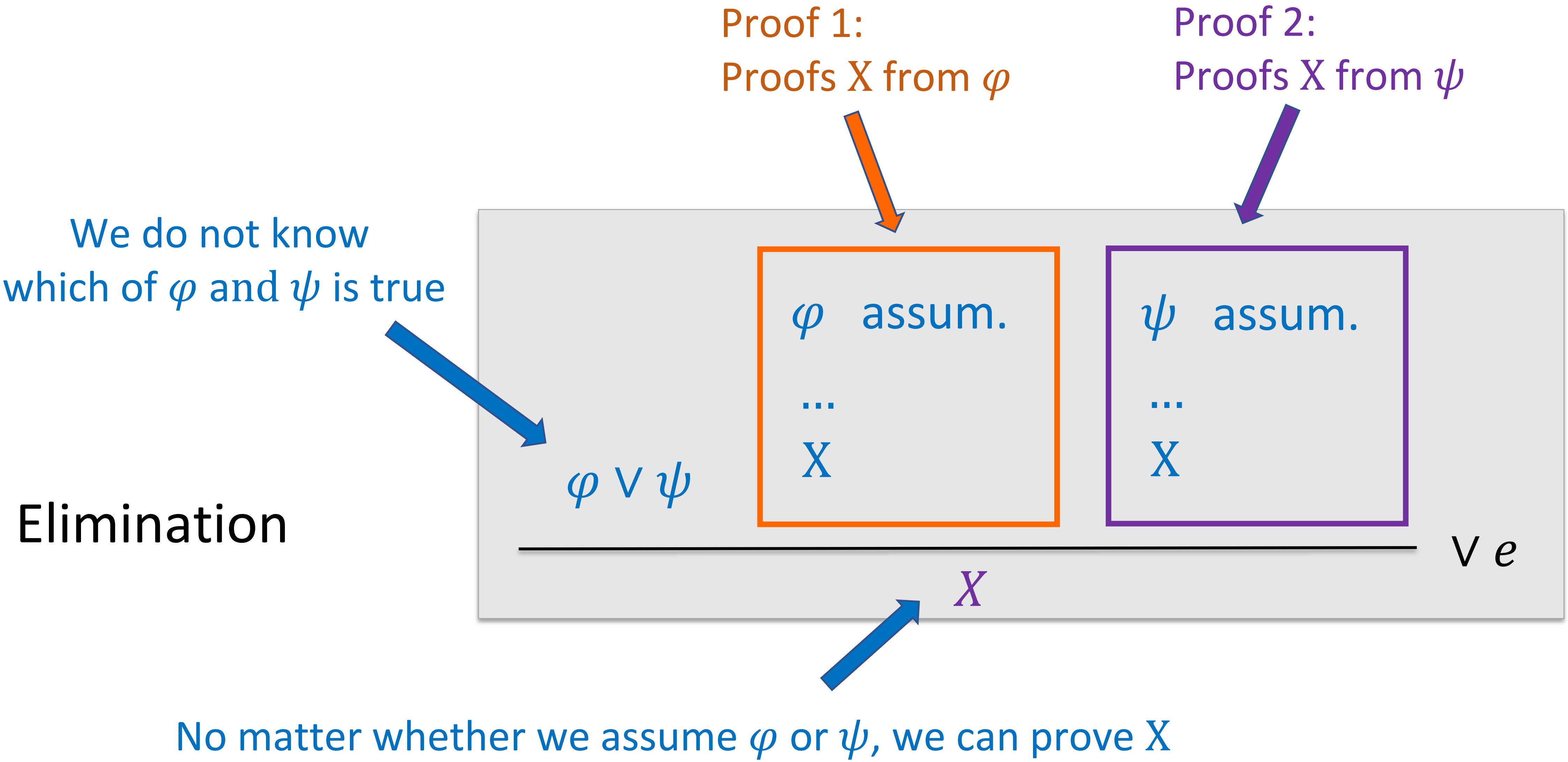
Rules for Disjunction

We do not know
which of φ and ψ is true

Elimination

$$\frac{\varphi \vee \psi}{X} \vee e$$

Rules for Disjunction



Rules for Disjunction

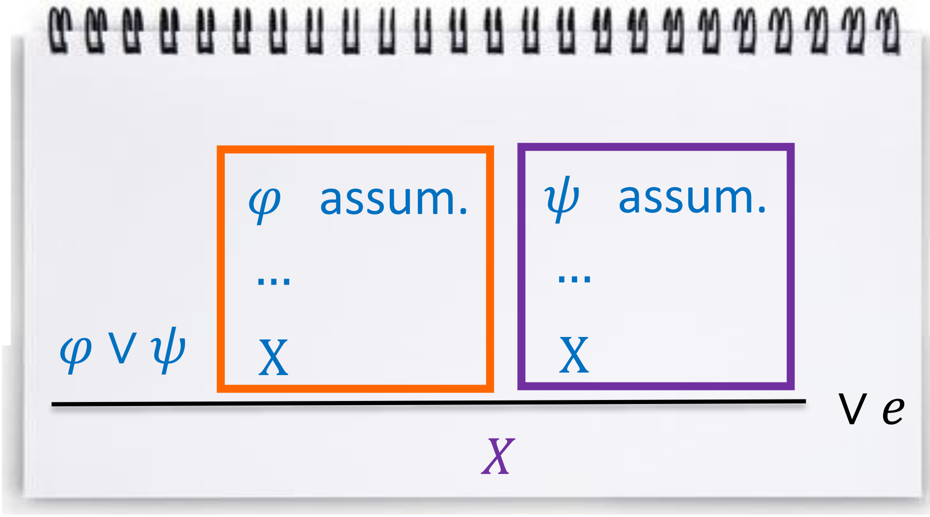
Introduction

$$\frac{\varphi}{\varphi \vee \psi} \vee i_1 \qquad \frac{\varphi}{\psi \vee \varphi} \vee i_2$$

Elimination

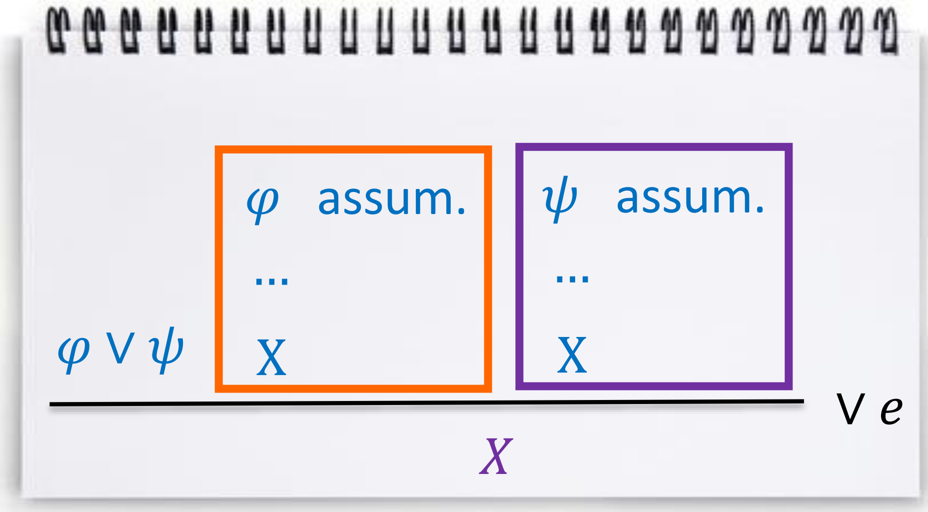
$$\frac{\varphi \vee \psi \quad \boxed{\begin{array}{l} \varphi \quad \text{assum.} \\ \dots \\ X \end{array}} \quad \boxed{\begin{array}{l} \psi \quad \text{assum.} \\ \dots \\ X \end{array}}}{X} \vee e$$

Example: $(p \wedge q) \vee (p \wedge r) \vdash \quad q \vee r$

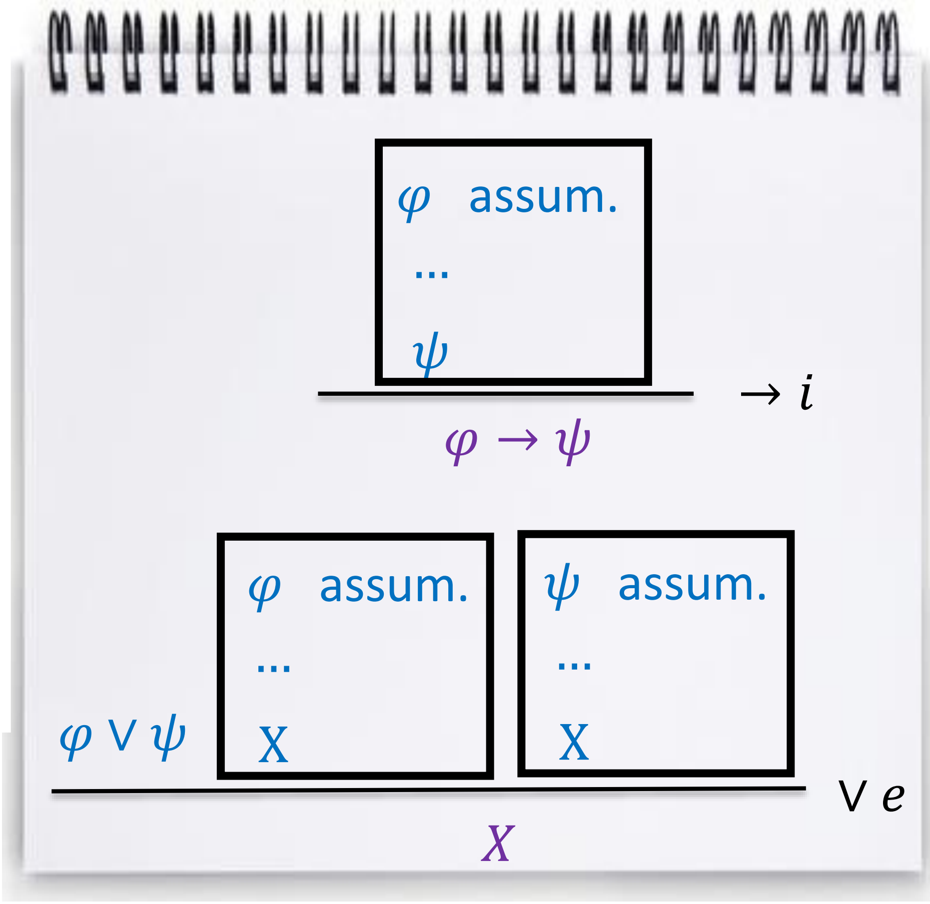


Example: $(p \wedge q) \vee (p \wedge r) \vdash \; q \vee r$

1.	$(p \wedge q) \vee (p \wedge r)$	premise
2.	$p \wedge q$	assumption
3.	q	$\wedge e_2$ 2
4.	$q \vee r$	$\vee i_1$ 3
5.	$p \wedge r$	assumption
6.	r	$\wedge e_2$ 5
7.	$q \vee r$	$\vee i_2$ 6
8.	$q \vee r$	$\vee e$ 1, 2 – 4, 5 – 7

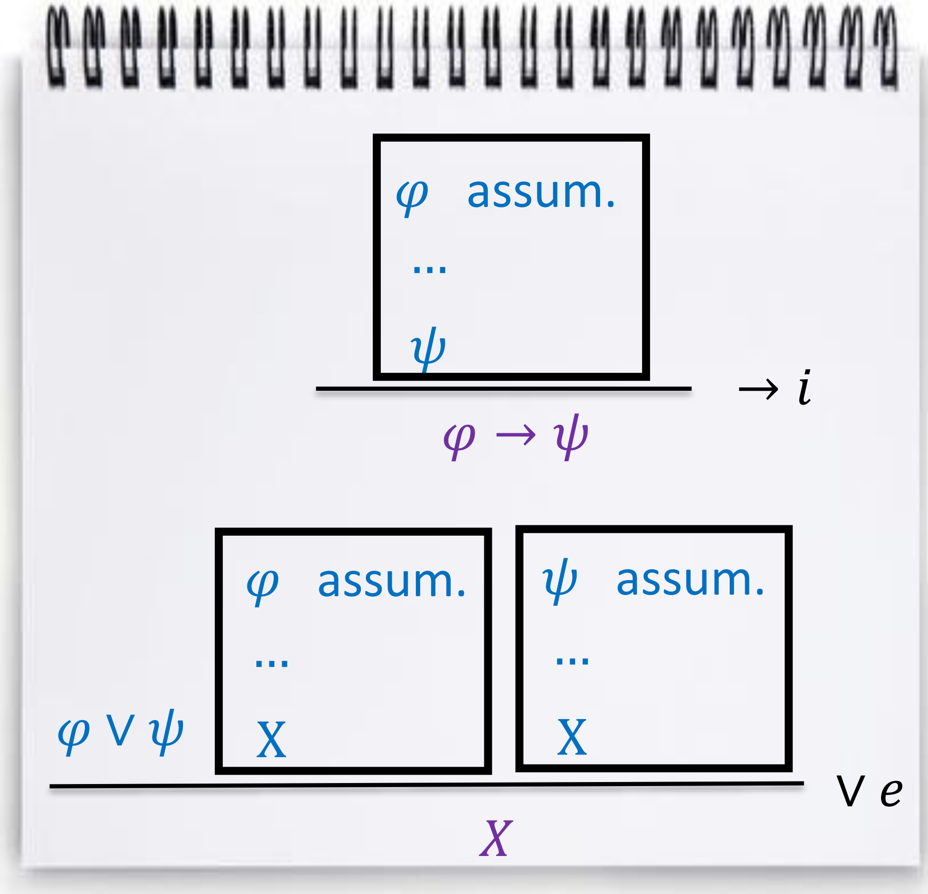


Example: $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$



Example: $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	prem.
2.	$p \lor q$	ass.
3.	p	ass.
4.	$p \lor r$	$\lor i_1$ 3
5.	q	ass.
6.	r	$\to$e 5,1
7.	$p \lor r$	$\lor$i 6
8.	$p \lor r$	$\lor$e 2,3-4,5-7
9.	$p \lor q \to (p \lor r)$	$\to$i 2-8



Rules for Negation

¬(Not) Elimination

$$\frac{\varphi \qquad \neg\varphi}{\bot} \; \neg e$$

Contradiction

Rules for Negation

¬(Not) Elimination

$$\frac{\varphi \qquad \neg\varphi}{\bot} \neg e$$

⊥(bottom) - Elimination

$$\frac{\bot}{\varphi} \bot e$$

A contradiction can prove anything.

Semantic Intuition:

- $p \vdash q$ ... whenever p is true, q must be true
- $p \land \neg p \vdash q$ ... $p \land \neg p$ is never true, no requirements on q

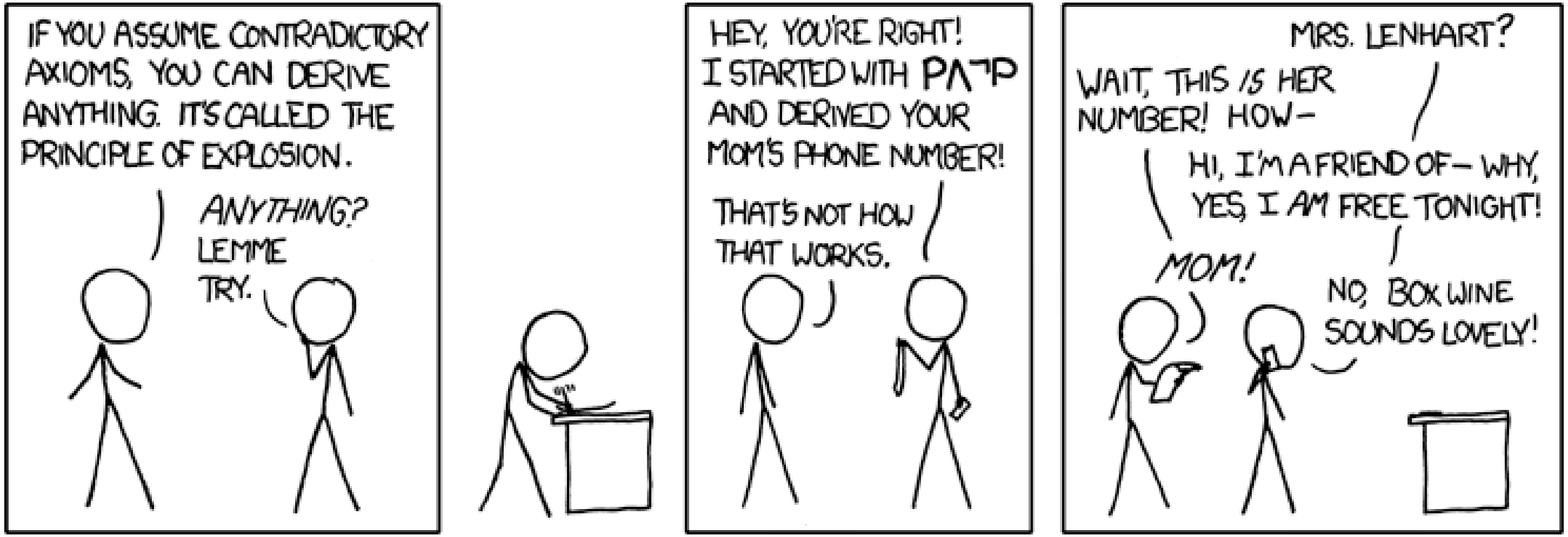
Rules for Negation

¬(Not) Elimination

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg e$$

⊥(bottom) - Elimination

$$\frac{\bot}{\varphi}\ \bot e$$



A contradiction can prove anything!

https://xkcd.com/704/

Example: $\neg p \vee q \vdash p \to q$

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg e$$

$$\frac{\bot}{\varphi}\ \bot e$$

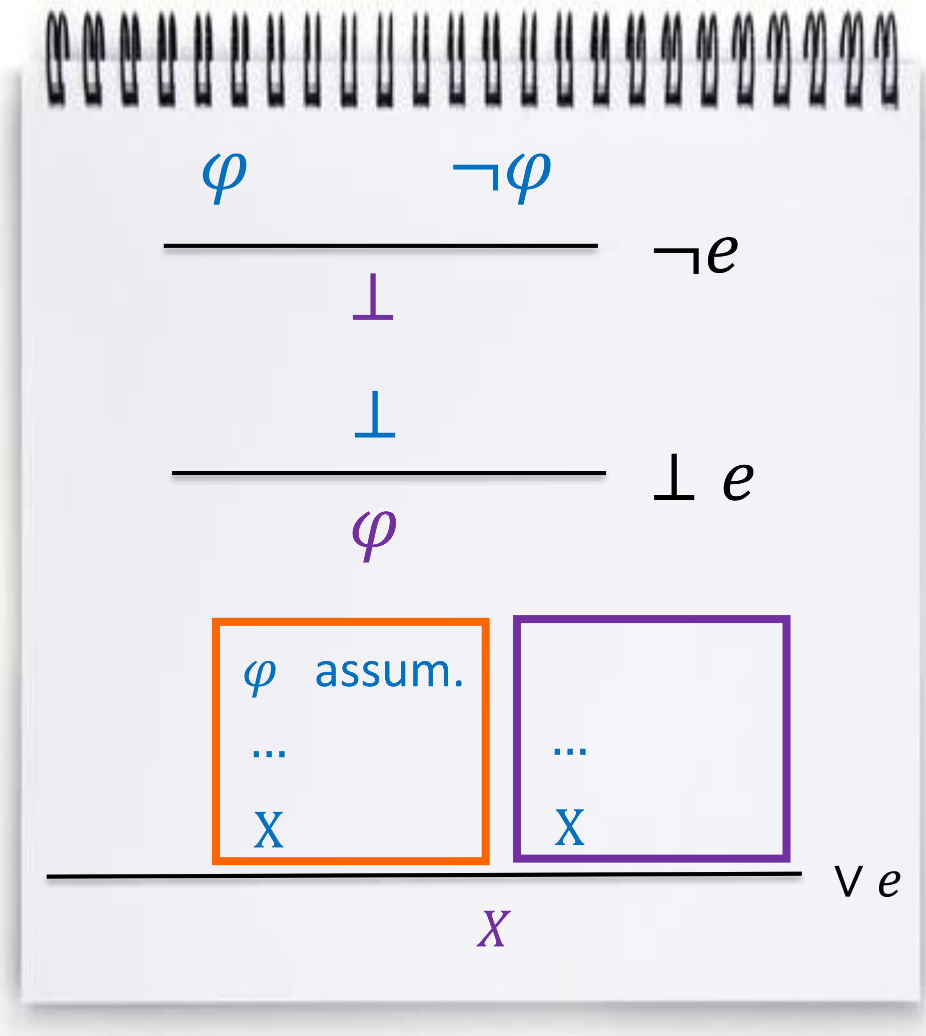
Example: $\neg p \vee q \vdash p \rightarrow q$

1	$\neg p \vee q$	premise
2	$\neg p$	assumption
3	p	assumption
4	$\bot$	$\neg$e 3, 2
5	q	$\bot$e 4
6	$p \rightarrow q$	$\rightarrow$i 3–5
7	q	assumption
8	p	assumption
9	q	copy 8
10	$p \rightarrow q$	$\rightarrow i\ 8-9$
11	$p \rightarrow q$	$\vee e\ 1, 2-6, 7-10$

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg e$$

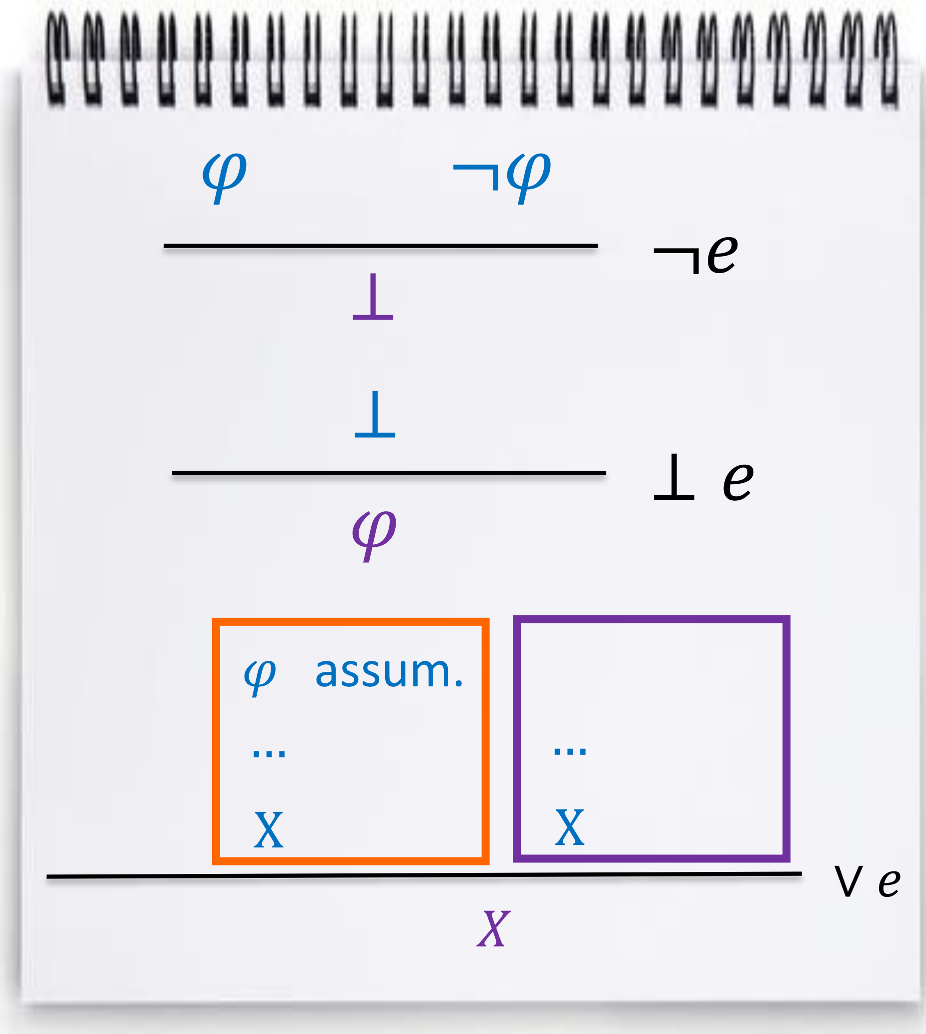
$$\frac{\bot}{\varphi}\ \bot e$$

Example $p \vee \neg\neg q, \neg p \wedge \neg q \vdash s \vee \neg t$



Example $p \lor \neg\neg q, \neg p \land \neg q \vdash s \lor \neg t$

1.	$p \lor \neg\neg q$	prem.
2.	$\neg p \land \neg q$	prem.
3.	p	ass.
4.	$\neg p$	$\land$e1 2
5.	$\bot$	$\neg$e 3,4
6.	$s \lor \neg t$	$\bot$e 5
7.	$\neg\neg q$	ass.
8.	$\neg q$	$\land$e2 2
9.	$\bot$	$\neg$e 7,8
10.	$s \lor \neg t$	$\bot$e 9
11.	$s \lor \neg t$	$\lor$e 1, 3-6, 7-10



Rules for Negation

¬ Introduction

φ	assum.
...	
$\perp$	

$$\frac{\boxed{\begin{array}{ll}\varphi & \text{assum.}\\ \dots & \\ \perp & \end{array}}}{\neg\varphi}\ \neg i$$

Assumption φ leads to contradiction.
Thus, assumption must be false.

- Hint
  - If it is of the form $\neg\varphi$, apply $\neg i$ immediately

Rules for Negation

¬Introduction

φ assum.

…

$\bot$

$\neg\varphi$ ¬i

Derived Rule - Proof by Contradiction

$\neg\varphi$ assum.

…

$\bot$

φ PBC

Other Rules

Law-of-the-Excluded-Middle Rule

$$\frac{}{\varphi \lor \neg\varphi}\ LEM$$

Afterwards, apply $\lor e$
Gives you a case split
→ Two simpler sub-proofs

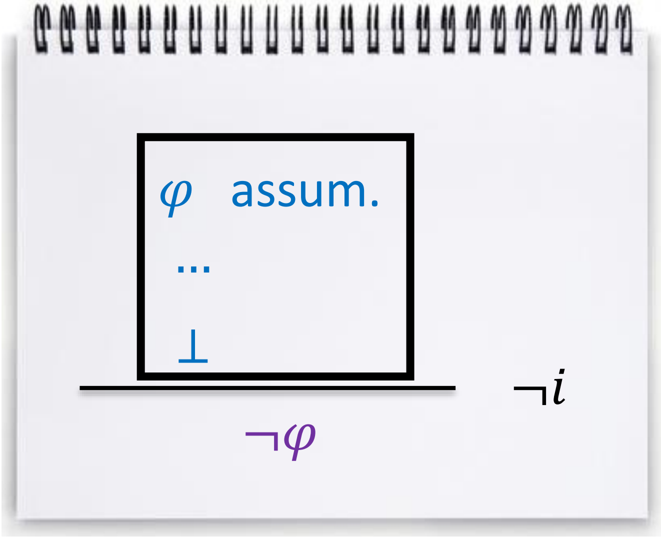
Copy-Rule

$$\frac{\varphi}{\varphi}\ copy$$

Use formulas already proven before.
Be careful with scopes of formulas.

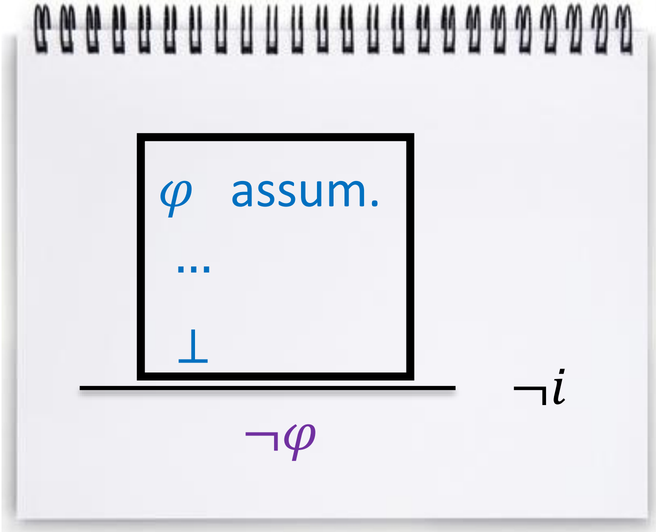
Example: $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	prem.
2.	q	prem.
3.		
4.		
5.		
6.	$\neg p$	

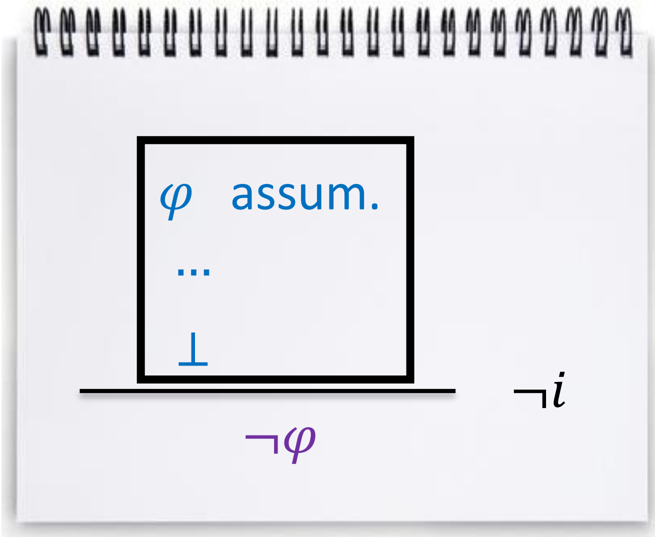


Example: $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	prem.
2.	q	prem.
3.	p	ass.
4.	$\neg q$	$\to$e 3,1
5.	$\bot$	$\neg$e 2,4
6.	$\neg p$	$\neg$i 3-5

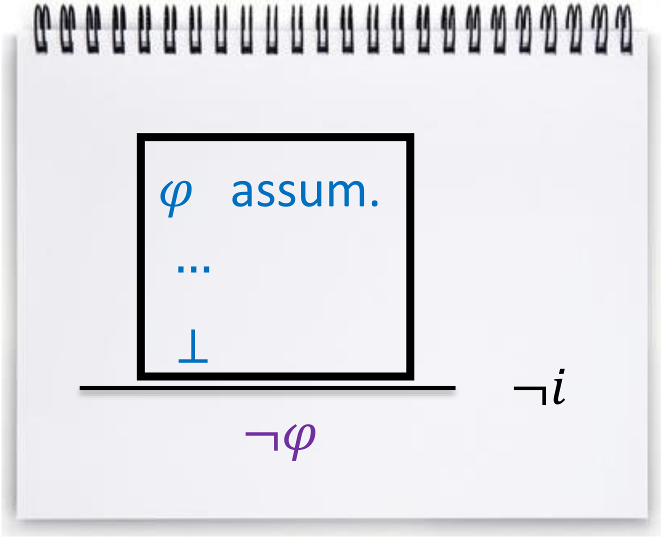


Example: $\neg q \vee \neg p \ \vdash \ \neg(q \wedge p)$



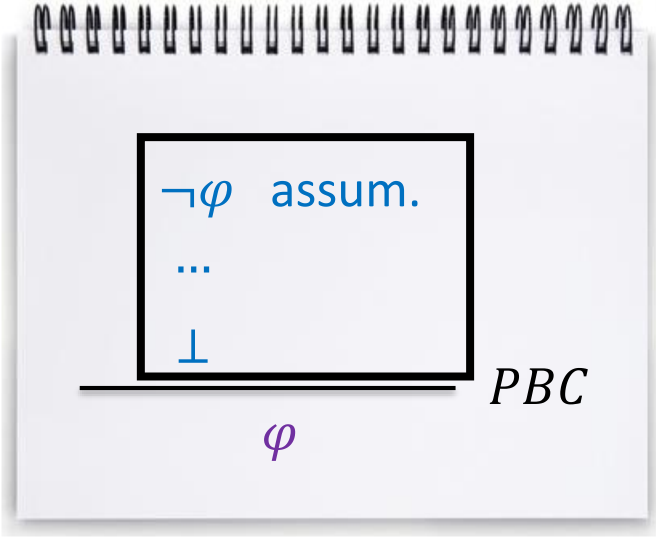
Example: $\neg q \vee \neg p \ \vdash \ \neg(q \wedge p)$

1.	$\neg q \vee \neg p$	prem.
2.	$q \wedge p$	ass.
3.	$\neg q$	ass.
4.	q	$\wedge$e1 2
5.	$\bot$	$\neg$e 3,4
6.	$\neg p$	ass.
7.	p	$\wedge$e2 2
8.	$\bot$	$\neg$e 6,7
9.	$\bot$	$\vee$e 1, 3-5, 6-8
10.	$\neg(q \wedge p)$	$\neg$i 2-9



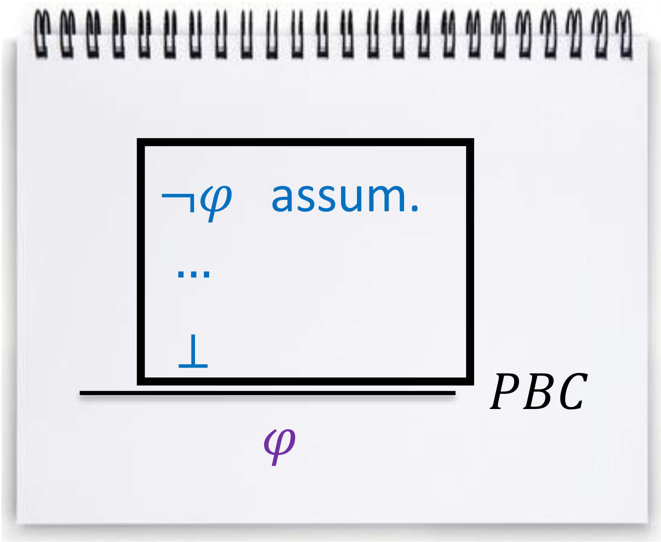
Example: $\neg p \rightarrow \neg q, q \vdash p$

1.	$\neg p \rightarrow \neg q$	prem.
2.	q	prem.
3.		
4.		
5.		
6.	p	



Example: $\neg p \rightarrow \neg q, q \vdash p$

1.	$\neg p \rightarrow \neg q$	prem.
2.	q	prem.
3.	$\neg p$	ass.
4.	$\neg q$	$\rightarrow$e 3,1
5.	$\bot$	$\neg$e 2,4
6.	p	PBC 3-5



Example: $\vdash p \to (q \to p)$

1.
2.
3.
4.
5. $p \to q \to p$

Example: $\vdash p \rightarrow (q \rightarrow p)$

1.	p	ass.
2.	q	ass.
3.	p	copy 1
4.	$q \rightarrow p$	$\rightarrow$i 2-3
5.	$p \rightarrow q \rightarrow p$	$\rightarrow$i 1-4

Soundness ("Korrektheit")

- Definition

$$\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vdash \psi} \quad \Rightarrow \quad \boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vDash \psi}$$

Correct **syntactic** entailment
From $\phi_1 \dots \phi_n$ we can **prove** that ψ holds

Correct **semantic** entailment
Each model that satisfies all premises $\phi_1 \dots \phi_n$ also satisfies ψ.

Therefore: $\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$ is valid

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Each model that satisfies all premises $\phi_1 \dots \phi_n$ also satisfies ψ.

Therefore: $\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$ is valid

- Meaning
  - Every provable sequent is a correct semantic entailment.
  - Semantically incorrect entailments are not provable.
    - $\phi_1, \phi_2, \dots, \phi_n \nvDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \nvdash \psi$

Completeness ("Vollständigkeit")

- Definition

$$\boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vDash \psi} \quad \Rightarrow \quad \boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vdash \psi}$$

Each model that satisfies all premises $\phi_1 \ldots \phi_n$ also satisfies ψ

From $\phi_1 \ldots \phi_n$ we can **prove** that ψ holds

- Meaning
  - Every correct semantic entailment has a proof.
  - Unprovable sequents are incorrect entailments.
    - $\phi_1, \phi_2, \ldots, \phi_n \nvdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \nvDash \psi$

Invalid Sequents

- How can we prove that there does not exists a proof for an invalid sequent?
  - E.g. $p \vee q \nvdash p \wedge q$

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    - $\phi_1, \phi_2, \ldots, \phi_n \nvDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \nvdash \psi$

Invalid Sequents

- How can we prove that there does not exists a proof for an invalid sequent?
  - E.g. $p \lor q \nvdash p \land q$

- Consequence of Soundness
  - Semantically incorrect entailments are not provable.
    - $\phi_1, \phi_2, \dots, \phi_n \nvDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \nvdash \psi$

- We need to find a model $\mathcal{M}$ that is a **counterexample**
- $\mathcal{M}$ is a counterexample if…
  - $\mathcal{M}$ satisfies all premises, and
  - $\mathcal{M}$ does not satisfy the conclusion

Invalid Sequents

- Find a counterexample to prove $p \lor q \nvdash p \land q$

Invalid Sequents

- Find a counterexample to prove $p \vee q \nvdash p \wedge q$

- Model $\mathcal{M}$: $p = T \quad q = F$
  - $\mathcal{M}$ satisfies all premises
    - $\mathcal{M} \vDash \mathrm{p} \vee q$ ✓

  - $\mathcal{M}$ does not satisfy the conclusion
    - $\mathcal{M} \nvDash \mathrm{p} \wedge q$ ✓

  - **Therefore, $\mathcal{M}$ is a counterexample!** $\mathcal{M}$ proves $\mathrm{p} \vee q \nvdash \mathrm{p} \wedge q$

Invalid Sequents

- Find a counterexample to prove $p \to q, q \to r \nvdash r$

Invalid Sequents

- Find a counterexample to prove $p \rightarrow q, q \rightarrow r \nvdash r$

- Model $\mathcal{M}$: $p = F \quad q = F \quad r = F$
  - $\mathcal{M}$ satisfies all premises
    - $\mathcal{M} \vDash p \rightarrow q$ and $\mathcal{M} \vDash q \rightarrow r$ ✓

  - $\mathcal{M}$ does not satisfy the conclusion
    - $\mathcal{M} \nvDash r$ ✓

- Therefore, $\mathcal{M}$ is a counterexample! $\mathcal{M}$ proves $p \rightarrow q, q \rightarrow r \nvdash r$

Tips for Deduction

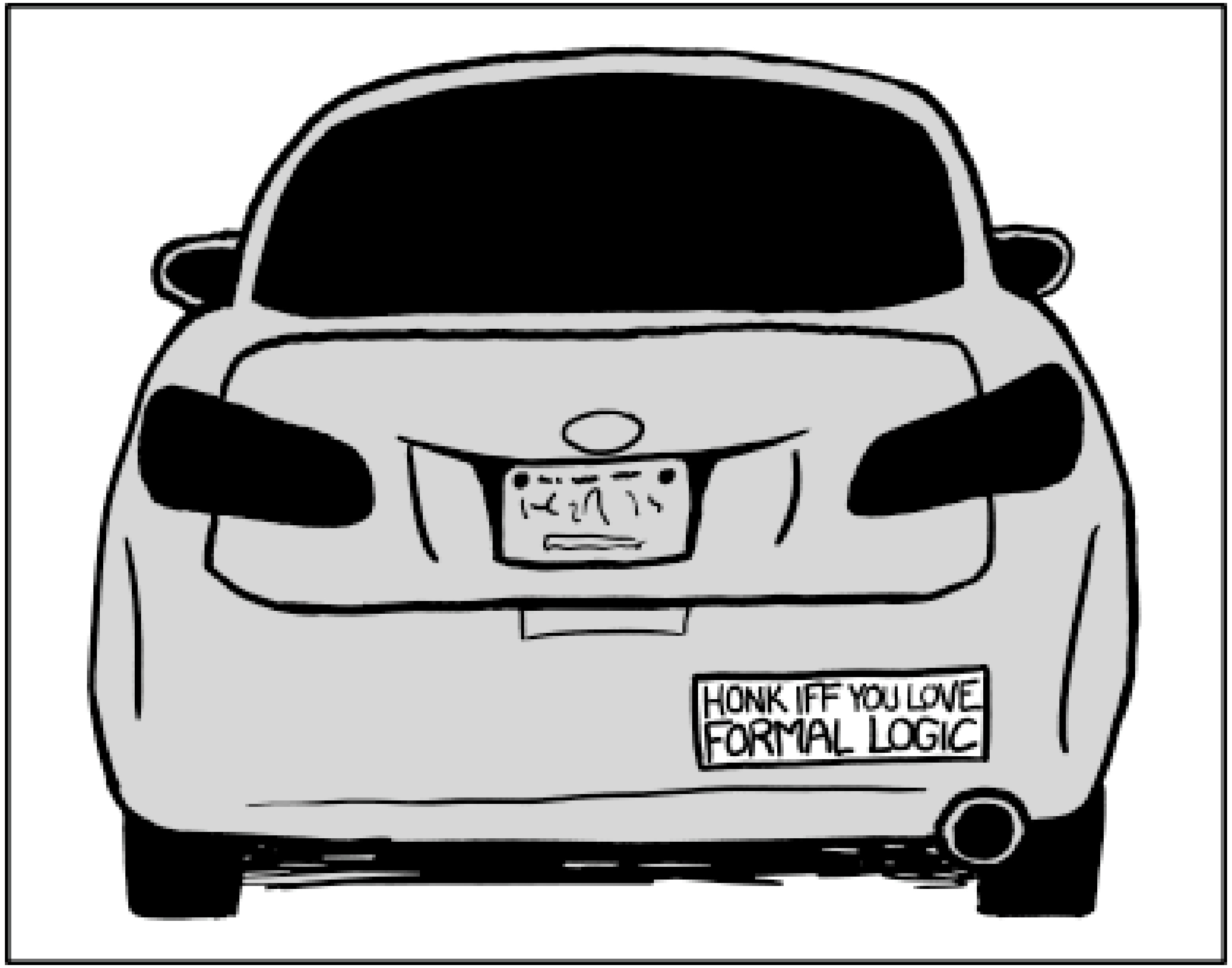
- Work from both sides
- Look at the conclusion
  - If it is of the form $\varphi \to \psi$, apply immediately $\to i$
  - If it is of the form $\neg\varphi$, apply immediately $\neg i$
- If you get stuck
  - Try case splits: LEM
  - Try proof by contradiction

Learning Outcomes

After this lecture…

1. students can explain the proof rules of ND for prop. logic.
2. students can construct ND proofs for valid sequents.
3. students can construct counterexamples for invalid sequents.
4. students can explain (a) what it means that ND for prop. logic is sound and complete and (b) can explain the consequences of it’s soundness and completeness.

Thank You



https://xkcd.com/1033/