Hardware Implementation of Public-Key Cryptography

Cryptography on Hardware Platform

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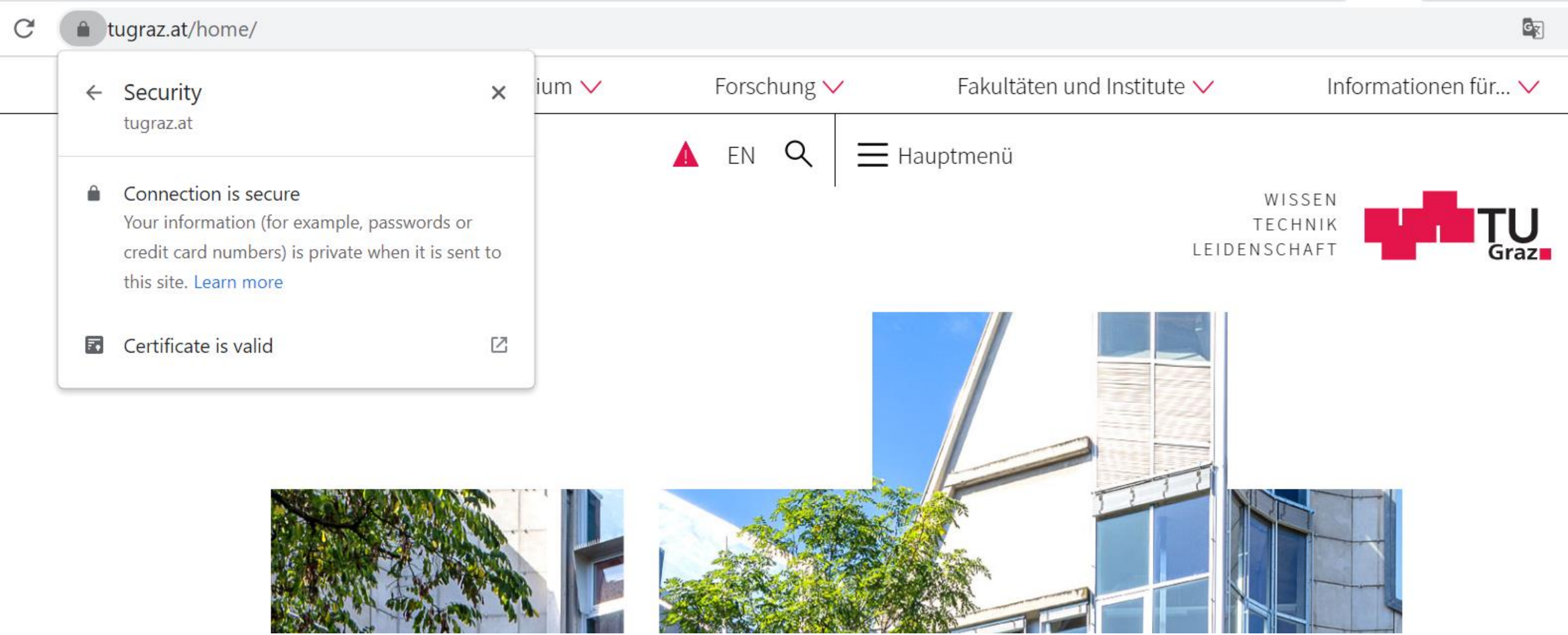
Outline

1. Public-key cryptography basics
2. Lattice-based public-key encryption
3. Polynomial arithmetic



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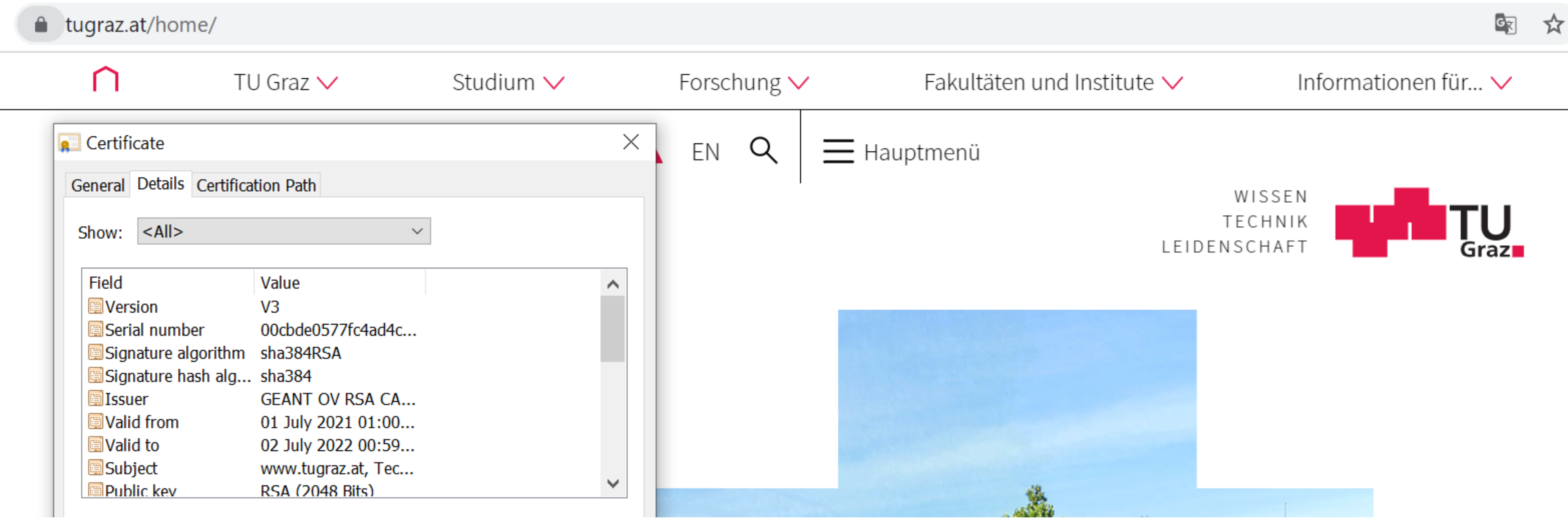
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Subject
www.tugraz.at, Tec...
Public key
RSA (2048 Bits)

Contemporary Cryptographic Primitives (examples)

Public-key Cryptography

- RSA
- Elliptic Curve

Symmetric-key Cryptography

- AES
- SHA-2 or SHA-3

Diffie-Hellman Key Agreement

Public info: Prime p and base g

Secret a

Secret b

$x = g^{a} \bmod p$

$y = g^{b} \bmod p$

Computes $y^{a} \bmod p$
$= g^{ab} \bmod p$

Computes $x^{b} \bmod p$
$= g^{ab} \bmod p$

Security is based on Discrete Log Problem (DLP)

Discrete Logarithm Problem

Given x, g and p, compute the secret a such that

$$x = g^a \bmod p$$

Latest record (Dec 2019) is 795-bit [BGGHTZ'19]
Using Intel Xeon Gold with 6130 CPUs.

Uses Number Field Sieve and takes 3100 core years using 1 CPU.

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Death of public key cryptography???

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The latest news from Gc

Quantum Supremacy Using a Programmable Superconducting Processor

Wednesday, October 23, 2019

Posted by John Martinis, Chief Scientist Quantum Hardware and Sergio Boixo, Chief Scientist Quantum Computing Theory, Google AI Quantum

to search

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) may

both display "quantum primacy" over classical computers

BY CHARLES Q. CHOI | 06 NOV 2021 | 2 MIN READ

Post Quantum Public Key Cryptography

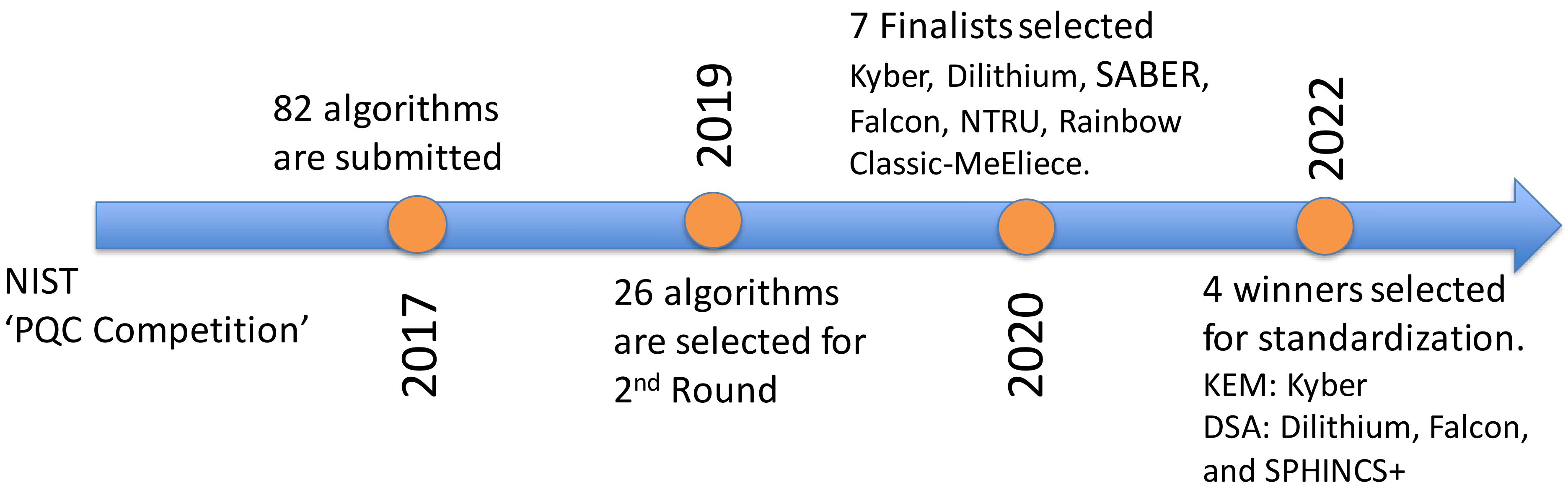
Post-quantum cryptographic (PQC) algorithms are designed using problems that are presumed to be unsolvable using quantum computers.

Currently 5 major problems are used for PQC.

- Lattice-based
- Code-based
- Multivariate-based
- Hash-based
- Isogeny-based

NIST Post Quantum Cryptography Standardization (2016-22)

NIST initiated PQC Standardization in 2016 and called for proposals.



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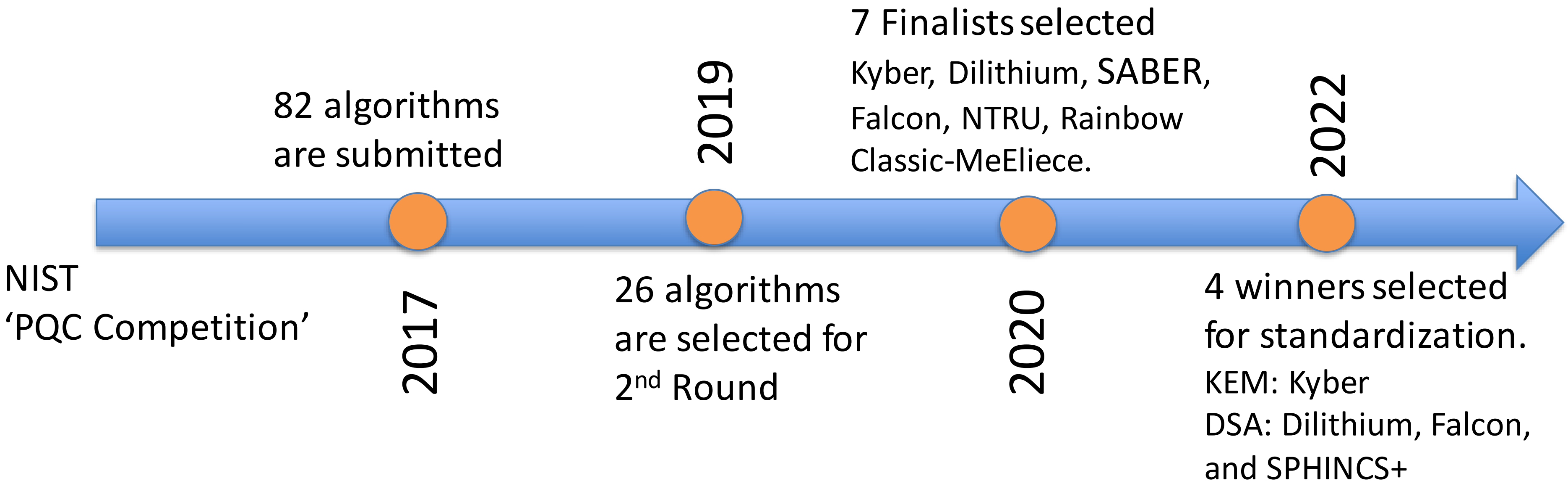


First three winners are all lattice-based. SPHINCS+ is hash-based.

NIST Post Quantum Cryptography Standardization (2022-)

To diversify portfolio of PQC algorithms, NIST called for additional PQC algorithms in 2022. There are around 40 new submissions.

- Code-based
  - Enhanced pqsigRM
  - FuLeeca
  - LESS
  - MEDS
  - Wave
- Isogenies
  - SQISign
- Lattices
  - EHT
  - EagleSign
  - HAETAE
  - HAWK
  - HuFu
  - Raccoon
  - Squirrels
- MPC-in-the-Head
  - CROSS
  - MIRA
  - MQOM
  - MiRitH
  - PERK
  - RYDE
  - SDitH
- Symmetric
  - AIMer
  - Ascon-Sign
  - FAEST
  - SPHINCS-alpha
- Multivariate
  - 3WISE
  - Biscuit
  - DME-Sign
  - HPPC
  - MAYO
  - PROV
  - QR-UOV
  - SNOVA
  - TUOV
  - UOV
  - VOX
- Other
  - ALTEQ
  - KAZ-Sign
  - PREON
  - Xifrat1-Sign.I
  - eMLE-Sig 2.0

Image source https://datatracker.ietf.org/meeting/117/materials/slides-117-tls-new-post-quantum-signature-algorithms-on-the-horizon-00

Outline

1. Public-key cryptography basics
2. **Lattice-based public-key encryption**
3. Polynomial arithmetic

In this course we will implement a simple lattice-based encryption scheme.

Lattice-based Cryptography – The LWE problem

Given two linear equations with unknown x and y

$$3x + 4y = 26$$
$$2x + 3y = 19$$

or

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 26 \\ 19 \end{pmatrix}$$

Find x and y.

Solving System of Linear Equations

For an unknown vector ***s*** of size n

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ \vdots \\ b_m \end{pmatrix}$$

Gaussian elimination solves ***s*** when *the* number of equations $m \geq n$

Solving System of Linear Equations after *Error* is added

$$\underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}}_{\text{Public } \mathbf{A}} \cdot \underbrace{\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}}_{\text{Secret } s} + \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \\ \vdots \\ e_m \end{pmatrix}}_{\text{Error } e} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ \vdots \\ b_m \end{pmatrix}}_{\text{Public } \mathbf{b}} \mod q$$

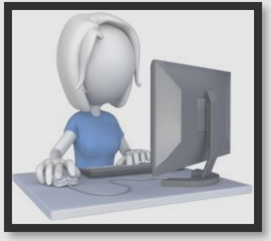
Learning With Errors (LWE) problem:

Given **(A, b)** → computationally infeasible to solve ***s***

Classical → Post-Quantum Diffie-Hellman key agreement

Public info: Prime p and base g

Secret a



$x = g^a \bmod p$ →

← $y = g^b \bmod p$

Secret b

Computes $y^a \bmod p = g^{ab} \bmod p$

Computes $x^b \bmod p = g^{ab} \bmod p$

Can we get a key agreement by replacing dLog with LWE problem?

LWE-based Diffie-Hellman Key-Exchange

Public uniformly random matrix **A** mod q

Small secret vector **[s]**
Small error vector **[e]**

$$b = A \times s + e$$

Small secret vector **[s']**
Small error vector **[e']**

$$b'^{T} = s'^{T} \times A + e'^{T}$$

$$v = b'^{T} \times s$$

Note: All operations are modulo *q*.

$$v' = s'^{T} \times b$$

Noisy **shared secret**

LWE-based Diffie-Hellman Key-Exchange (2)

What to do with the two ‘noisy’ integers?

$$v = b'^{T} \times s$$

$$v' = s'^{T} \times b$$

LWE-based Diffie-Hellman Key-Exchange (2)

What to do with the two ‘noisy’ integers?

This integer I is the same on both sides

v = Integer I + Noise E_1

v’ = Integer I + Noise E_2

E_1 and E_2 are quite small noise elements.

Most significant bit of v and v’ are equal with high probability → You get one key bit.

Ring-LWE problem

Given

$$a(x) * s(x) + e(x) = b(x) \pmod{q} \pmod{f(x)}$$

in a polynomial ring $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ where

$a(x)$: uniformly random public polynomial

$s(x)$: small secret polynomial

$e(x)$: small error polynomial

$b(x)$: output polynomial,

Ring-LWE problem:

Given $(a(x), b(x))$ → computationally infeasible to solve $s(x)$

$$f(x) = x^4 + 1$$

$$\begin{bmatrix} 1 & -4 & -3 & -2 \\ 2 & 1 & -4 & -3 \\ 3 & 2 & 1 & -4 \\ 4 & 3 & 2 & 1 \end{bmatrix}_{4\times 4} \times \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 - 24 - 21 - 16 \\ 10 + 6 - 28 - 24 \\ 15 + 12 + 7 - 32 \\ 20 + 8 + 14 + 8 \end{bmatrix} = \begin{bmatrix} -56 \\ -36 \\ 2 \\ 60 \end{bmatrix}$$

$$-32x^2 - 52x - 61 + 60x^3 + 34x^2 + 16x + 5$$

$$= -56 - 36x + 2x^2 + 60x^3$$

$$a(x) = 1 + 2x + 3x^2 + 4x^3$$

$$b(x) = 5 + 6x + 7x^2 + 8x^3$$

Ring-LWE-based Diffie-Hellman Key-Exchange

Public polynomial a(x)

Small secret poly s(x)

Small error poly e(x)

Small secret poly s’(x)

Small error poly e’(x)

$b(x) = a(x) \cdot s(x) + e(x)$

$b'(x) = a(x) \cdot s'(x) + e'(x)$

$v(x) = b'(x) \cdot s(x)$
$= a(x) \cdot s(x) \cdot s'(x) + e'(x) \cdot s(x)$

$v'(x) = b(x) \cdot s'(x)$
$= a(x) \cdot s(x) \cdot s'(x) + e(x) \cdot s'(x)$

Decoding v(x) gives n bits.

Decoding v’(x) gives n bits.

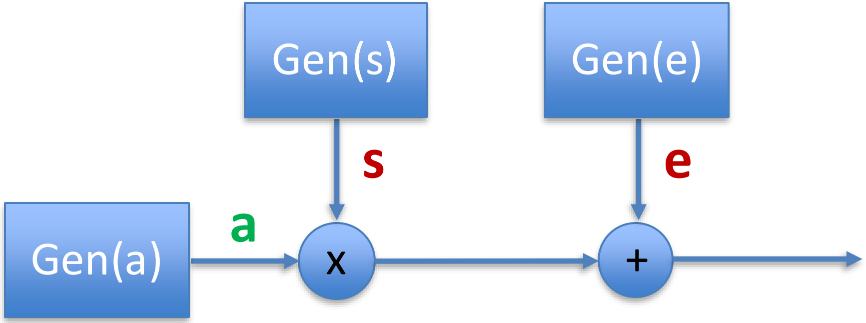
This course: Hardware implementation of Ring-LWE encryption

Ring-LWE (i.e., polynomials) is significantly more efficient than matrix LWE

Assignment 1: We implement ring-LWE public-key encryption (PKE)

Ring LWE-based Public-Key Encryption (PKE)

- **Key Generation:**
  - **Output:** public key (pk), secret key (sk)



b = a*s + e (Ring LWE sample)

Arithmetic operations are performed in a polynomial ring R_q

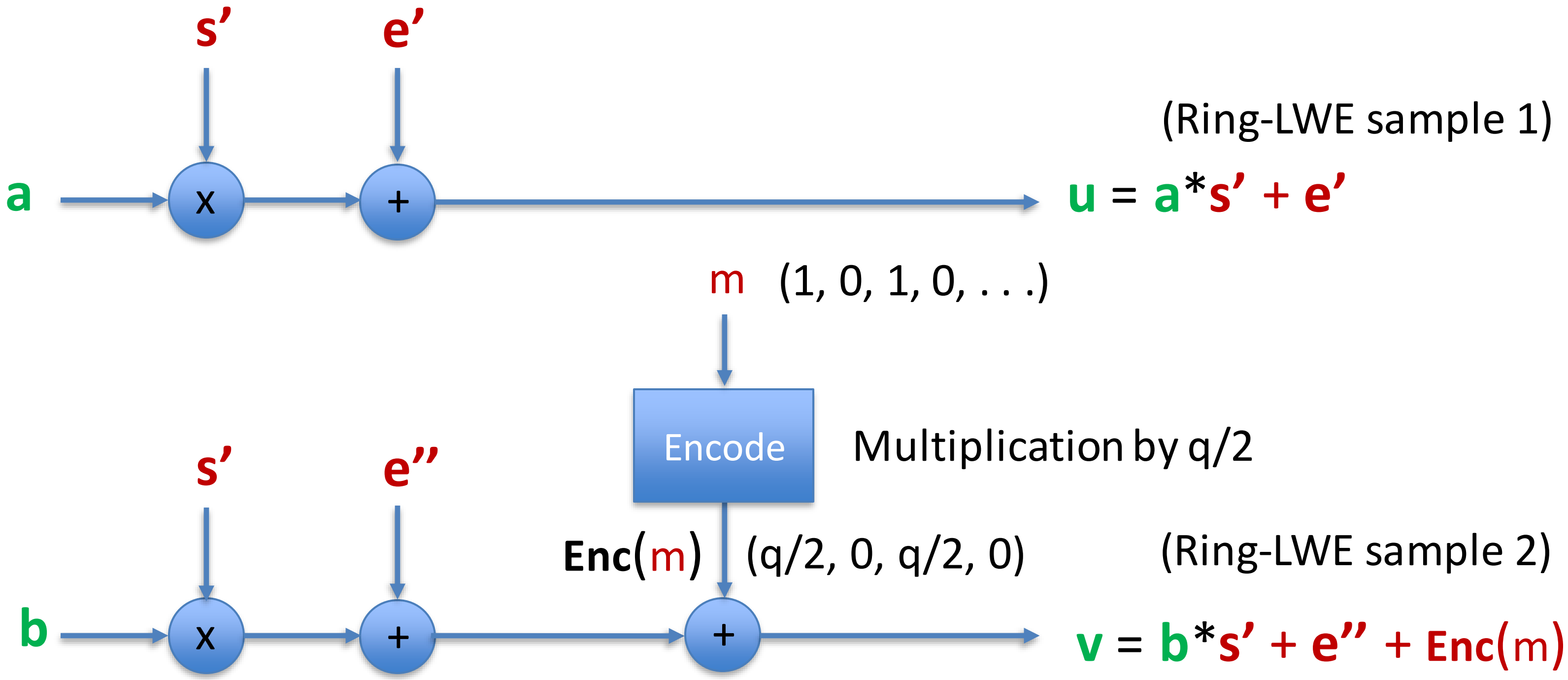
Public Key (pk): (a,b)

Secret Key (sk): (s)

V. Lyubashevsky, C. Peikert, and O. Regev. "On Ideal Lattices and Learning with Errors Over Rings". IACR ePrint 2012/230.

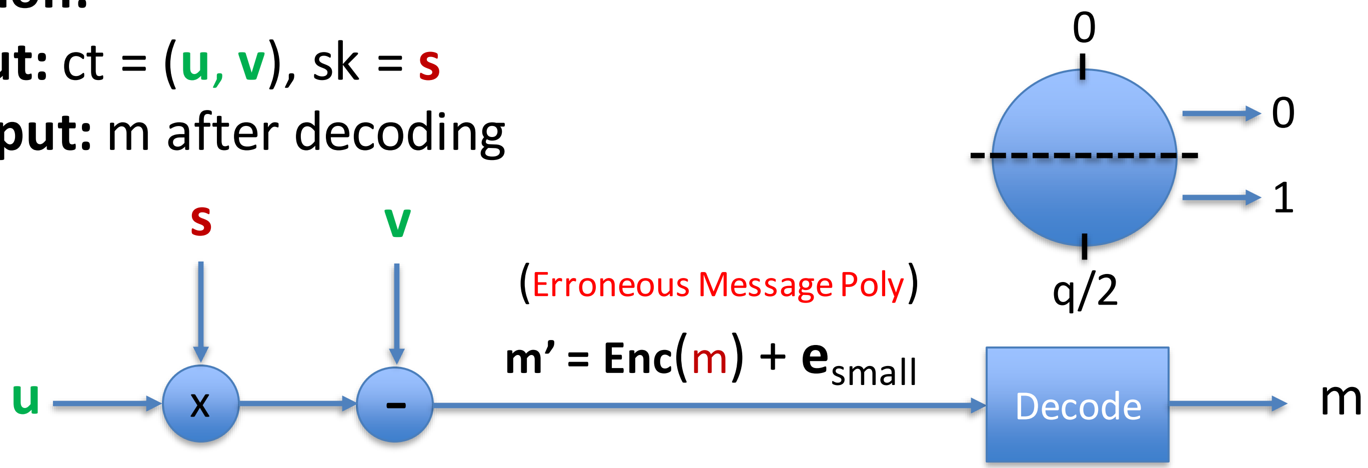
Ring LWE-based Public-Key Encryption (PKE)

- **Encryption:**
  - **Input: pk** = (a,b), message m
  - **Output: ct** = (u,v)



Ring LWE-based Public-Key Encryption (PKE)

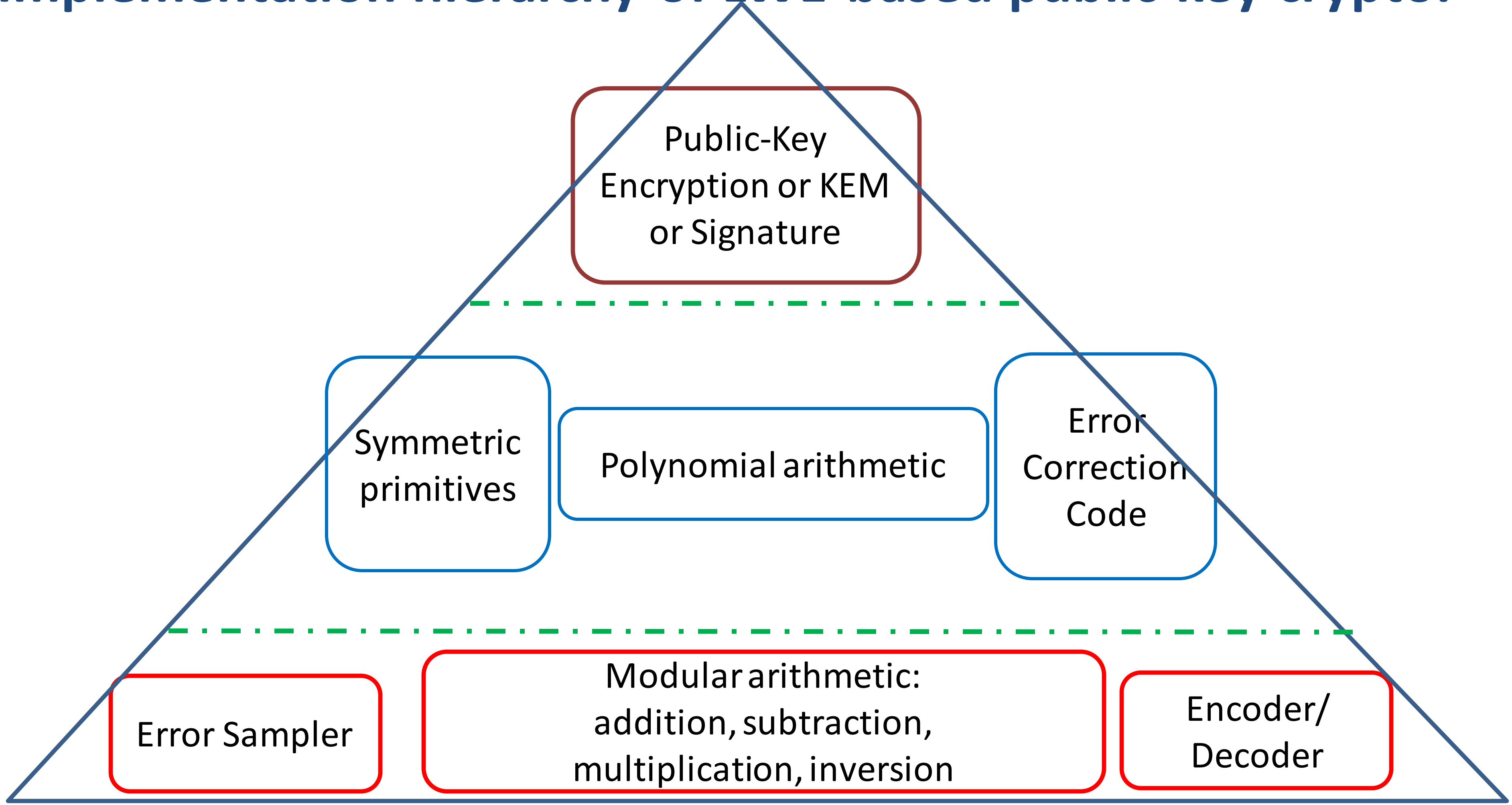
- **Decryption:**
  - **Input:** ct = (u, v), sk = s
  - **Output:** m after decoding



$$\mathbf{v} - \mathbf{u} * \mathbf{s} = \mathbf{m}' = \mathbf{Enc}(m) + (\mathbf{e} * \mathbf{s}' + \mathbf{e}'' + \mathbf{e}' * \mathbf{s})$$
$$= \mathbf{Enc}(m) + \mathbf{e}_{small}$$

Select most significant bit of each coefficient as the message bits

Implementation hierarchy of LWE-based public-key crypto.



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Mathematical background on Polynomial Arithmetic

Polynomial addition modulo q

Two polynomials are added coefficient-wise modulo q.

Example:

$$
\begin{array}{rl}
 & a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7 \\
+ & b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7 \\
\hline
\end{array}
$$

Polynomial addition modulo q

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+ & b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7 \\
\hline
 & c(x) = 1x^3 + 6x^2 + 0x + 1 \pmod 7
\end{array}
$$

Polynomial multiplication modulo *q*

Usual way: Multiply each term in one polynomial by each term in the other polynomial and then sum them following the standard way.

$$a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7$$

$$*$$

$$b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7$$

Polynomial multiplication modulo *q*

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$$3x^3 + 1x^2 + 4x + 5$$

Polynomial multiplication modulo *q*

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 & a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7 \\
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\hline
 & 3x^3 + 1x^2 + 4x + 5 \\
 & 4x^4 + 6x^3 + 3x^2 + 2x
\end{array}
$$

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$$3x^3 + 1x^2 + 4x + 5$$

$$4x^4 + 6x^3 + 3x^2 + 2x$$

$$3x^5 + 1x^4 + 4x^3 + 5x^2$$

Polynomial multiplication modulo *q*

Usual way: Multiply each term in one polynomial by each term in the other polynomial and then sum them following the standard way.

$$a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7$$

$$*$$

$$b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7$$

$$3x^3 + 1x^2 + 4x + 5$$

$$4x^4 + 6x^3 + 3x^2 + 2x$$

$$3x^5 + 1x^4 + 4x^3 + 5x^2$$

$$1x^5 + 5x^5 + 6x^4 + 4x^3$$

Polynomial multiplication modulo *q*

Usual way: Multiply each term in one polynomial by each term in the other polynomial and then sum them following the standard way.

$$a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7$$

$$* \quad b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7$$

$$3x^3 + 1x^2 + 4x + 5$$

$$4x^4 + 6x^3 + 3x^2 + 2x$$

$$3x^5 + 1x^4 + 4x^3 + 5x^2$$

$$1x^5 + 5x^5 + 6x^4 + 4x^3$$

$$c(x) = 1x^6 + 1x^5 + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

Coefficient-wise addition mod 7

Modular reduction of a polynomial by a polynomial

Let's say, we want to modulo reduce this polynomial

$$c(x) = 1x^6 + 1x^5 + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

by the following polynomial

$$f(x) = x^4 + 1 \pmod 7.$$

Modular reduction of a polynomial by a polynomial

Let’s say, we want to modulo reduce this polynomial

$$c(x) = 1x^6 + 1x^5 + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

by the following polynomial

$$f(x) = x^4 + 1 \pmod 7.$$

Any term in c(x) with degree ≥ deg(f) will get reduced by f(x) using the congruence relation:

$$x^4 = -1 \pmod 7$$

Modular reduction of a polynomial by a polynomial

Let's say, we want to modulo reduce this polynomial

$$c(x) = 1x^6 + 1x^5 + \boxed{4x^4} + 3x^3 + 2x^2 + 6x + 5 \pmod{7}$$

by the following polynomial

$$f(x) = x^4 + 1 \pmod{7}.$$

Any term in c(x) with degree ≥ deg(f) will get reduced by f(x) using the congruence relation:

$$x^4 = -1 \pmod{7}$$

Example:

$$4x^4 = 4\cdot(-1) \pmod{7}$$
$$= 3 \pmod{7}$$

Modular reduction of a polynomial by a polynomial

Let's say, we want to modulo reduce this polynomial

$$c(x) = \boxed{1x^6} + \boxed{1x^5} + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

by the following polynomial

$$f(x) = x^4 + 1 \pmod 7.$$

Any term in c(x) with degree ≥ deg(f) will get reduced by f(x) using the congruence relation:

$$x^4 = -1 \pmod 7$$

Similarly, $1x^5 = 6x \pmod 7$

and $1x^6 = 6x^2 \pmod 7$

Modular reduction of a polynomial by a polynomial

Let's say, we want to modulo reduce this polynomial

$$c(x) = 1x^6 + 1x^5 + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

by the following polynomial

$$f(x) = x^4 + 1 \pmod 7.$$

After reduction by $f(x)$

$$6x^2 + 6x + 3$$

Hence, $c(x) \bmod f(x) = (6x^2 + 6x + 3) + (3x^3 + 2x^2 + 6x + 5)$

$$= 3x^3 + 1x^2 + 5x + 1 \pmod 7 \pmod f$$

[Definition] Polynomial ring $R_q = \mathbb{Z}_q[x]/<f(x)>$

- The polynomial ring has its irreducible polynomial $f(x)$ of degree n.
  → Hence all ring-elements are polynomials of degree n-1.

- Closed under polynomial addition and multiplication.
  →For two polynomials $a(x)$ and $b(x)$ $\in R_q$

$$c(x) = a(x) + b(x) \pmod{q} \pmod{f} \in R_q$$

and

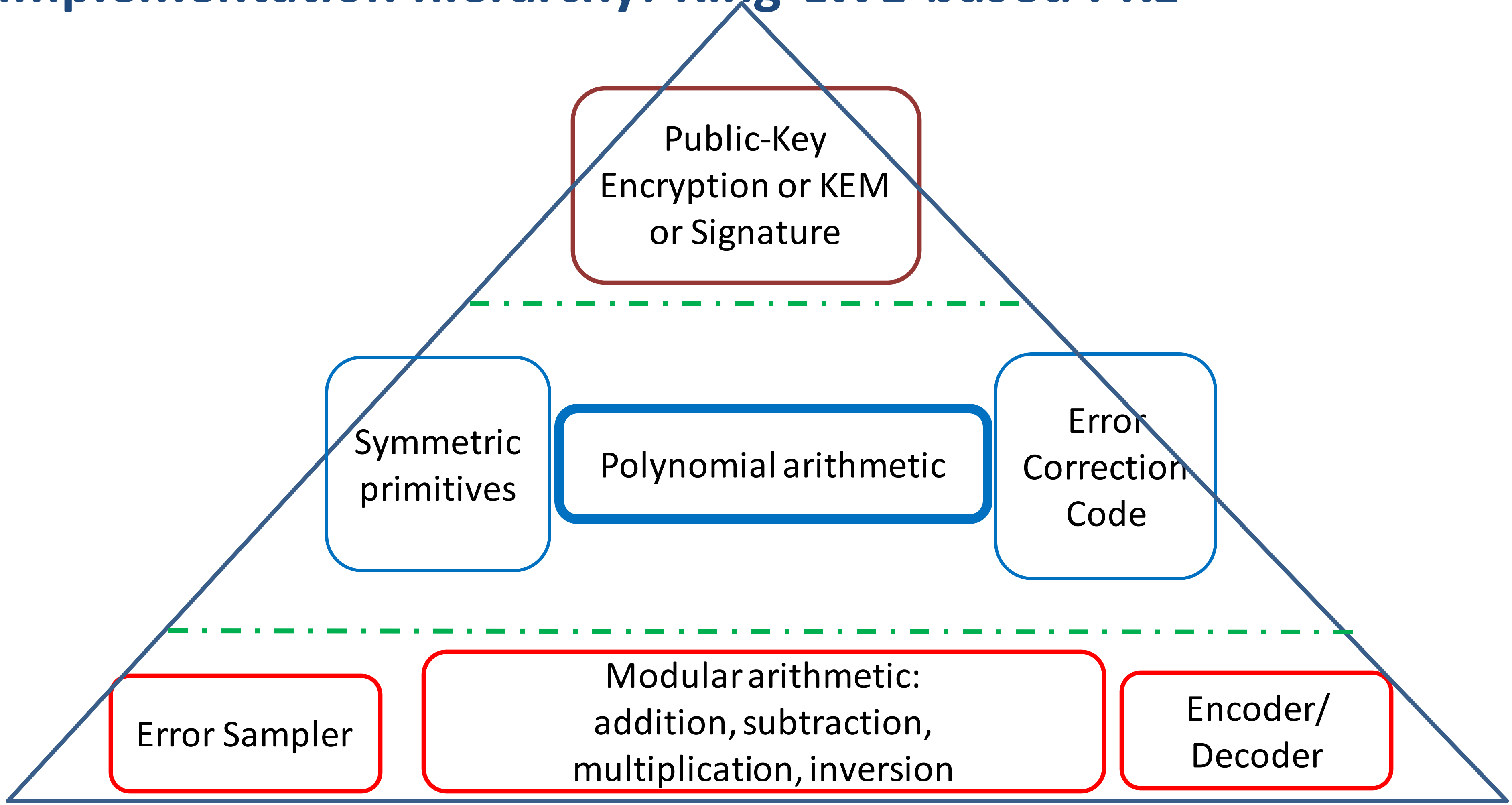
$$c(x) = a(x) * b(x) \pmod{q} \pmod{f} \in R_q$$

- Identity element under the addition rule is the 0-polynomial.
- Identity element under the multiplication rule is the 1-polynomial
- Multiplicative inverse of a polynomial may not exist.

From now on we assume all multiplications are in $R_q = \mathbb{Z}_q[x]/\langle x^n + 1\rangle$

→ This simplifies modular reduction by $f(x) = x^n + 1$
→ and makes an implementation more efficient

Implementation hierarchy: Ring-LWE-based PKE



How to multiply two polynomials?

We can use the following algorithms and also combinations of them

- Schoolbook multiplication: $O(n^2)$
- Karatsuba multiplication: $O(n^{1.585})$
- Fast Fourier Transform (FFT) multiplication: $O(n \log n)$

Schoolbook method of polynomial multiplication

$$
\begin{array}{rr}
 & a(x) = 5x^3 + 4x^2 + 2x + 6 \pmod 7 \\
* & b(x) = 3x^3 + 2x^2 + 5x + 2 \pmod 7 \\
\hline
 & 3x^3 + 1x^2 + 4x + 5 \\
 & 4x^4 + 6x^3 + 3x^2 + 2x \\
 & 3x^5 + 1x^4 + 4x^3 + 5x^2 \\
 & 1x^5 + 5x^5 + 6x^4 + 4x^3 \\
\hline
\end{array}
$$

$$c(x) = 1x^6 + 1x^5 + 4x^4 + 3x^3 + 2x^2 + 6x + 5 \pmod 7$$

We learnt this method during algebra classes in school.

+ Simple structure makes it easy to implement.

- Time complexity is $O(n^2)$, which is the worst of all three algorithms.

GP/Pari code for Schoolbook polynomial multiplication (1)

```
N = 2^8;     /* Polynomial degree */
q = 7681;  /* Coefficient modulus */
firr = Mod(1, q)*x^N + Mod(1, q);  /* Irreducible polynomial modulus */

schoolbook(a, b) = {

   /* Schoolbook polynomial multiplication  c = a*b has two nested loops */
   c = 0;

      for(i=0, N-1,
         for(j=0, N-1,
            mval = polcoeff(b, j)*polcoeff(a,i) % q;
            c = c + mval*x^(j+i)));

   c = c%firr;

   return (c);
}
```

https://pari.math.u-bordeaux.fr/gp.html

GP/Pari code for Schoolbook polynomial multiplication (2)

```
test() = {
   /* Formation of random polynomial a(x) with coefficients mod q */
   a = 0;
   for(i=0, N-1, a = a + random(q)*x^i);

   /* Formation of random polynomial b(x) with coefficients mod q */
   b = 0;
   for(i=0, N-1, b = b + random(q)*x^i);

   c= schoolbook(a, b);

   /* Native polynomial multiplication d = a*b.    */
   d = a*b % firr;


   print("c = ", c);
   print("d = ", d);
   print("c-d = ", c-d);        /* If correct, then c-d will be 0. */
}

test();
```

https://pari.math.u-bordeaux.fr/gp.html

Architecture for Schoolbook polynomial multiplication

E.g., polynomial degree $N = 256$ and $f(x) = x^{256} + 1$.

Algorithm: Schoolbook algorithm

$acc(x) \leftarrow 0$

for $i = 0; i < 256; i$++ **do**

  for $j = 0; j < 256; j$++ **do**

    $acc[j] = acc[j] + b[j] \cdot a[i]$

  $b = b \cdot x \bmod \langle x^{256} + 1 \rangle$

return acc

How will you implement the algo as an architecture in HW?

Architecture for Schoolbook polynomial multiplication

E.g., polynomial degree $N = 256$ and $f(x) = x^{256} + 1$.

Algorithm: Schoolbook algorithm

$acc(x) \leftarrow 0$

for $i = 0; i < 256; i{+}{+}$ **do**

  for $j = 0; j < 256; j{+}{+}$ **do**

    $acc[j] = acc[j] + b[j] \cdot a[i]$

  $b = b \cdot x \bmod \langle x^{256} + 1 \rangle$

return acc

How will you implement the algo as an architecture in HW?

- What are the fundamental elementary operations?

Architecture for Schoolbook polynomial multiplication

E.g., polynomial degree $N = 256$ and $f(x) = x^{256} + 1$.

Algorithm: Schoolbook algorithm

$acc(x) \leftarrow 0$

for $i = 0; i < 256; i{+}{+}$ **do**

  for $j = 0; j < 256; j{+}{+}$ **do**

    $acc[j] = acc[j] + b[j] \cdot a[i]$

  $b = b \cdot x \bmod \langle x^{256} + 1 \rangle$

return acc

Multiply and Accumulate (MAC)

How will you implement the algo as an architecture in HW?

- What are the fundamental elementary operations?
- Draw an architecture for MAC

Architecture for Schoolbook polynomial multiplication

E.g., polynomial degree $N = 256$ and $f(x) = x^{256} + 1$.

Algorithm: Schoolbook algorithm

$acc(x) \leftarrow 0$

for $i = 0; i < 256; i{+}{+}$ **do**

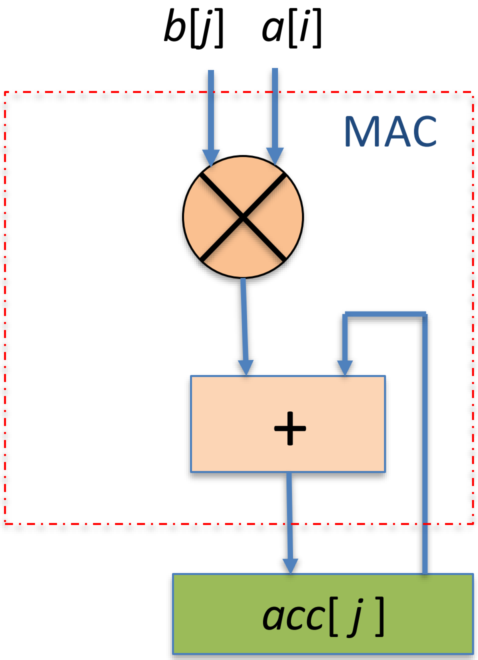
  for $j = 0; j < 256; j{+}{+}$ **do**

    $acc[j] = acc[j] + b[j] \cdot a[i]$

  $b = b \cdot x \bmod \langle x^{256} + 1 \rangle$

return acc

Architecture of MAC unit



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How to implement this step?

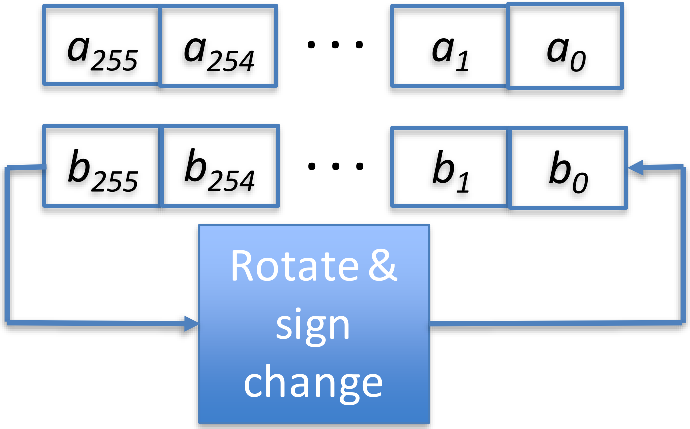
With mod $f(x) = x^n + 1$, we have $x^n \equiv -1$, hence multiplying

$b(x) = b_{n-1}x^{n-1} + \ldots + b_0 \pmod{f(x)}$ by x gives

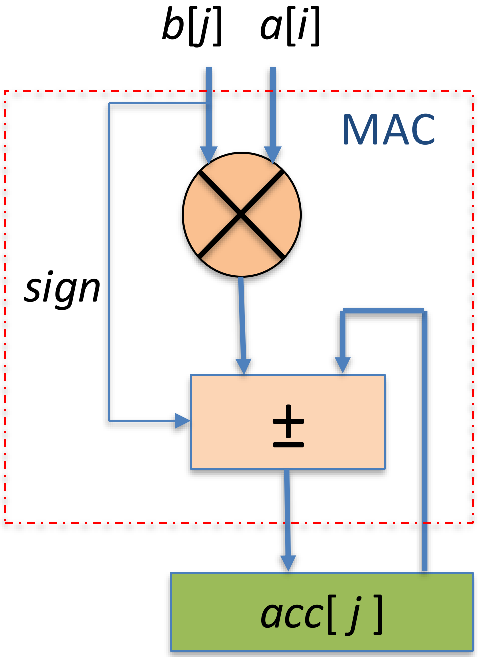
$x \cdot b(x) = b_{n-2}x^{n-1} + \ldots + b_0 x - b_{n-1} \pmod{f(x)}$ →Rotation with sign change.

Architecture for Schoolbook polynomial multiplication

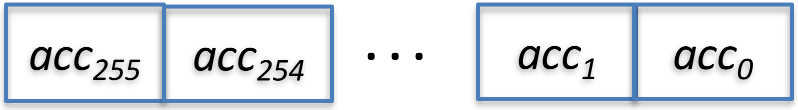
Ring-buffer registers



Apply this MAC( ) one by one.



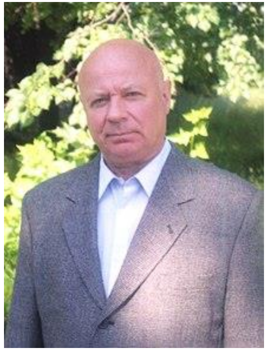
Note: This is just an idea. This may **not** be an optimized architecture!



Karatsuba method of polynomial multiplication

Andrey Kolmogorov
(1903-1987)

In 1960, during a seminar at Moscow State University, Kolmogorov conjectured that multiplying two integers have $O(n^2)$ complexity.



Anatoly Karatsuba
(1937-2008)

Karatsuba, then a 23 years old student, attended the seminar and within a week came up with a divide-and-conquer method for multiplying two integers with $O(n^{\log_2 3})$ complexity.

The method was published in the Proceedings of the USSR Academy of Sciences in 1962.

Karatsuba method of polynomial multiplication (1)

Split each operand into two halve-size polynomials:

$$a(x) = \underbrace{a_{n\text{-}1}\, x^{n\text{-}1} + \;\ldots\; + a_{n/2}\, x^{n/2}}_{a_h(x)} + \underbrace{a_{n/2\text{-}1}\, x^{n/2\text{-}1} + \ldots\; + a_1 x + a_0}_{a_l(x)}$$

Hence, we can write:

$$a(x) = a_h(x)\, x^{n/2} + a_l(x) = a_h x^{n/2} + a_l$$

Karatsuba method of polynomial multiplication (2)

After splitting we have:

$$a(x) = a_h x^{n/2} + a_l$$

$$b(x) = b_h x^{n/2} + b_l$$

Naïve method: We can compute the result using the *Schoolbook* method

$$a(x) * b(x) = a_h b_h x^n + (a_h b_l + a_l b_h) x^{n/2} + a_l b_l$$

It performs 4 multiplication and has a quadratic complexity.

Karatsuba showed how to compute this using 3 multiplications.

Karatsuba method of polynomial multiplication (3)

After splitting we have:

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Karatsuba method:

$$a(x) * b(x) = a_h b_h x^n + (a_h b_l + a_l b_h) x^{n/2} + a_l b_l$$

It computes $(a_h b_l + a_l b_h)$ term by performing only one multiplication as:

$$(a_h b_l + a_l b_h) = (a_h + a_l)\cdot(b_h + b_l) - \underbrace{a_h b_h - a_l b_l}$$

These two produces are reused from the above.

Karatsuba method of polynomial multiplication (3)

After splitting we have:

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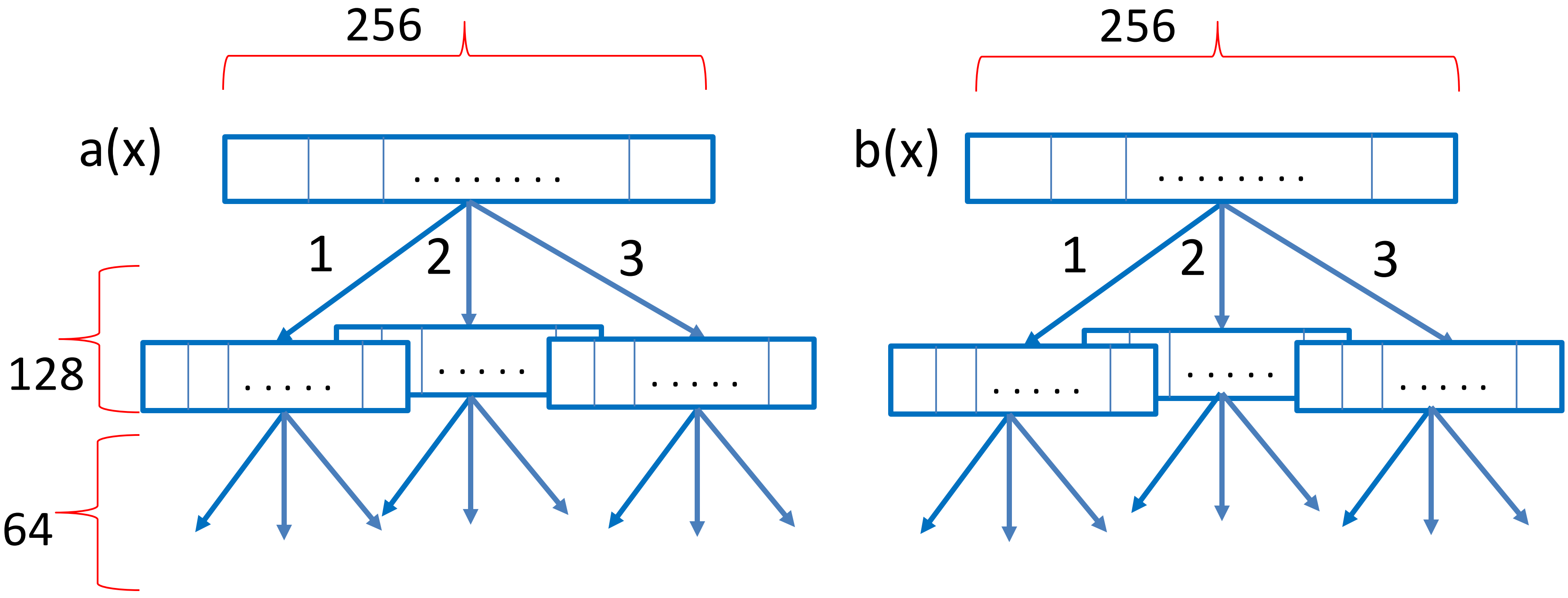
Hence, the three multiplications are:
$a_h b_h$, $a_l b_l$, and $(a_h + a_l) \cdot (b_h + b_l)$.

Divide-and-Conquer approach: Karatsuba tree



- Recursively apply divide-and-conquer strategy
- When the polynomials are of sufficiently-small size, multiply them
- And return to the higher levels

Complexity of Karatsuba polynomial multiplication

Let, T_n be the time for multiplication two *n*-coefficient polynomials.

$$
\begin{aligned}
T_n &= 3T_{n/2} \\
&= 3^2 T_{n/4} \\
&= 3^3 T_{n/8} \\
&= \ldots \\
&= 3^{\log_2 n} T_1
\end{aligned}
$$

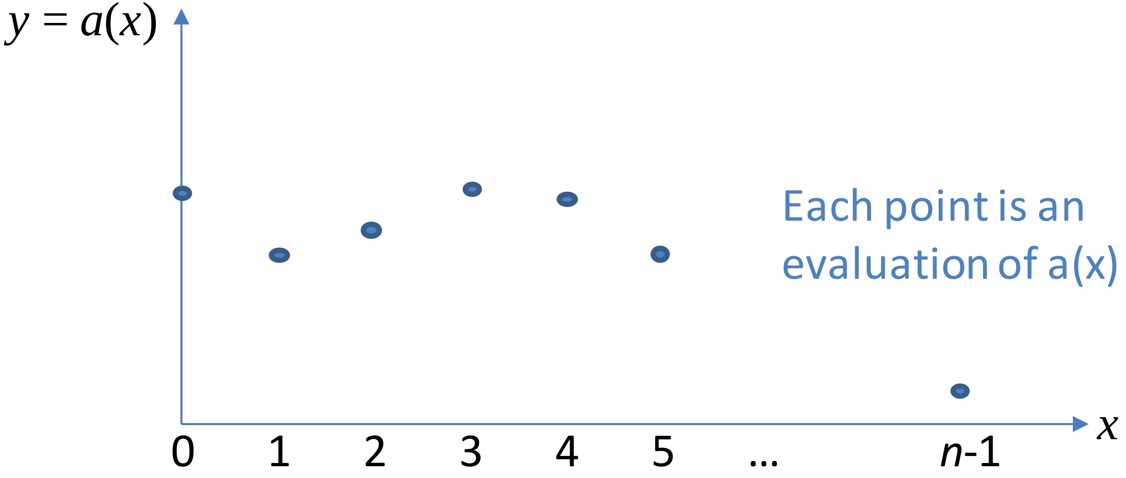
Hence, the complexity = $O(3^{\log_2 n}) = O(n^{\log_2 3}) \approx O(n^{1.585})$

The idea of FFT

Representation: Polynomial ↔ Point values

Given a polynomial a(x) we can easily compute its evaluations at *n* points

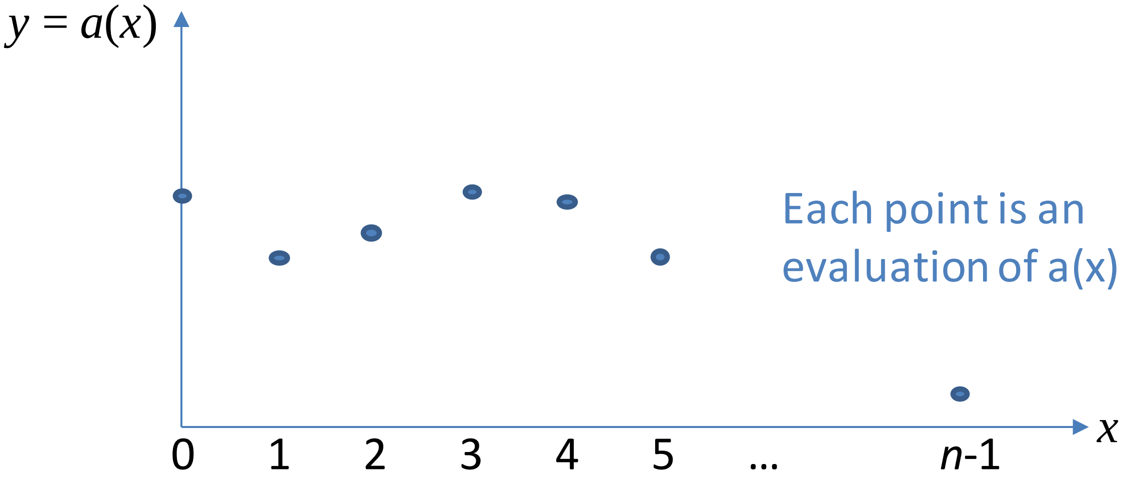
$$a(x) = a_{n\text{-}1}\, x^{n\text{-}1} + \ldots + a_1 x + a_0$$



Representation: Polynomial ↔ Point values

Given *n* distinct evaluation points $y_0, y_1, \ldots, y_{n-1}$ can we get $a(x)$?

$$a(x) = ?$$



Representation: Polynomial ⟷ Point values

What we have as $y_0, y_1, \ldots, y_{n-1}$ are:

$$y_0 = a(0) = a_{n-1}\, 0^{n-1} + \ldots + a_2 0^2 + a_1 0 + a_0$$

$$y_1 = a(1) = a_{n-1}\, 1^{n-1} + \ldots + a_2 1^2 + a_1 1 + a_0$$

$$\ldots$$

$$y_{n-1} = a(n-1) = a_{n-1}\, (n-1)^{n-1} + \ldots + a_2(n-1)^2 + a_1(n-1) + a_0$$

$$\underbrace{\begin{bmatrix} 0^0 & 0^1 & 0^2 & \ldots & 0^{n-1} \\ 1^0 & 1^1 & 1^2 & \ldots & 1^{n-1} \\ 2^0 & 2^1 & 2^2 & \ldots & 2^{n-1} \\ & & \ldots & & \end{bmatrix}}_{V} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \ldots \end{bmatrix} = \begin{bmatrix} a(0) \\ a(1) \\ a(2) \\ \ldots \end{bmatrix}$$

(V^{-1} performs the opposite)

Polynomial → Point values

$$\begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \dots \\ a(n\text{-}1) \end{pmatrix} = \begin{pmatrix} 0^0 & 0^1 & 0^2 & \dots & 0^{n\text{-}1} \\ 1^0 & 1^1 & 1^2 & \dots & 1^{n\text{-}1} \\ 2^0 & 2^1 & 2^2 & \dots & 2^{n\text{-}1} \\ & & \dots & & \\ (n\text{-}1)^0 & & & & (n\text{-}1)^{n\text{-}1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_{n\text{-}1} \end{pmatrix}$$

Points

Polynomial coefficients

Given a polynomial, calculating the *n* distinct points is called ‘evaluation’.

Point values → Polynomial

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_{n-1} \end{bmatrix} = \begin{pmatrix} 0^0 & 0^1 & 0^2 & \dots & 0^{n-1} \\ 1^0 & 1^1 & 1^2 & \dots & 1^{n-1} \\ 2^0 & 2^1 & 2^2 & \dots & 2^{n-1} \\ & & \dots & & \\ (n-1)^0 & & & & (n-1)^{n-1} \end{pmatrix}^{-1} \begin{bmatrix} a(0) \\ a(1) \\ a(2) \\ \dots \\ a(n-1) \end{bmatrix}$$

Polynomial coefficients

Points

Given n distinct points, calculating the polynomial is called ‘interpolation’.

Rules: Polynomial ↔ Point values

1. Interpolation will succeed in obtaining a(x) only if there are n distinct evaluations $y_0, \ldots, y_{n-1}$.

2. You can choose any values for x as long as you get n distinct y_i.

Application of DFT in polynomial multiplication

$$a(x) = a_0 + a_1 x + \ldots + a_{n\text{-}1} x^{n\text{-}1}$$
$$b(x) = b_0 + b_1 x + \ldots + b_{n\text{-}1} x^{n\text{-}1} \qquad \times$$

$$c(x) = a(x)*b(x) = c_0 + c_1 x + \ldots + c_{n\text{-}1} x^{n\text{-}1} + \ldots + c_{2n\text{-}2} x^{2n\text{-}2}$$

Polynomial c(x) has degree 2*n*-2.

→ Therefore c(x) can be represented as 2*n*-1 discrete points.

Application of DFT in polynomial multiplication

- For c(x) = a(x) * b(x) where a(x) and b(x) have degree of n-1:
  - Evaluate a(x) and b(x) at 2n-1 points
  - Multiply evaluated points $m_i = a(i).b(i)$
  - Use Lagrange's interpolating polynomials to reconstruct c(x)

$$c(x) = a(x) * b(x) = \sum_{i=0}^{2n-2} i.\, L_i(x) \text{ where } L_i(x) = \prod_{i \neq j} \frac{x-j}{i-j}$$

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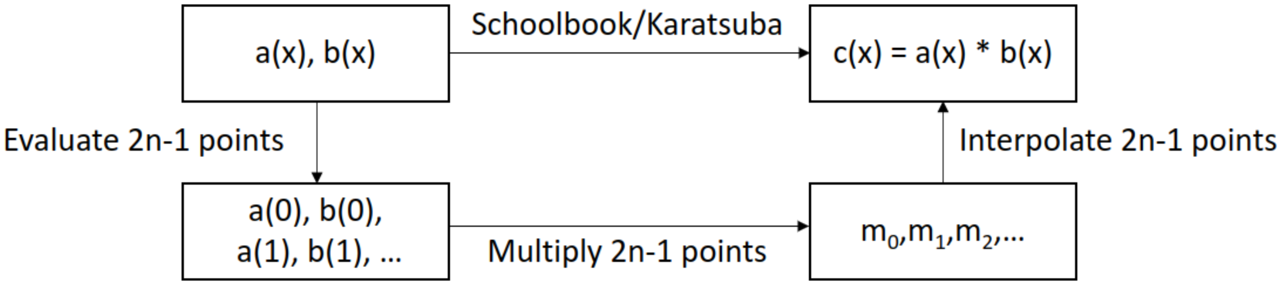

Application of DFT in polynomial multiplication

- Observation: If we can perform evaluation and interpolation operations fast, then we can multiply two polynomials fast.
  - Can we use DFT to perform these operations?

Application of DFT in polynomial multiplication

- Observation: If we can perform evaluation and interpolation operations fast, then we can multiply two polynomials fast.
  - Can we use DFT to perform these operations?

- Discrete Fourier Transform (DFT)
  - A transformation $(a_0, a_1, \ldots, a_{n-2}, a_{n-1})$ --> $(A_0, A_1, \ldots, A_{n-2}, A_{n-1})$

$$A_k = \sum_{j=0}^{n-1} a_j . e^{\left(-\frac{(2i\pi)}{n}\right).k.j}$$

  - $\omega = e^{-i2\pi/n}$ is n-th primitive root of 1 (unity) which satisfies $\omega^n = 1$

$\omega^k \neq 1$ for $1 \leq k < n$

Application of DFT in polynomial multiplication

- We can choose our evaluation points as powers of ω

$$\underbrace{\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \dots & \omega^{0\text{-}1} \\ \omega^0 & \omega^1 & \omega^2 & \dots & \omega^{n-1} \\ \omega^0 & \omega^2 & \omega^4 & \dots & \omega^{2n-2} \\ & & \dots & & \\ & & & & \end{bmatrix}}_{V(\omega)} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \\ \dots \end{bmatrix} = \begin{bmatrix} a(\omega^0) \\ a(\omega^1) \\ a(\omega^2) \\ \\ \dots \end{bmatrix}$$

$$V(\omega) * V(\omega^{-1}) = n * I$$

$$V(\omega^{-1}) = n * V(\omega)^{-1}$$

$$V(\omega)^{-1} = (1/n) * V(\omega^{-1})$$

With V(ω) (**DFT**), we compute *evaluation*

With V(ω)-1 or (1/n) * $V(\omega^{-1})$ (**IDFT**), we compute *interpolation*

Application of DFT in polynomial multiplication

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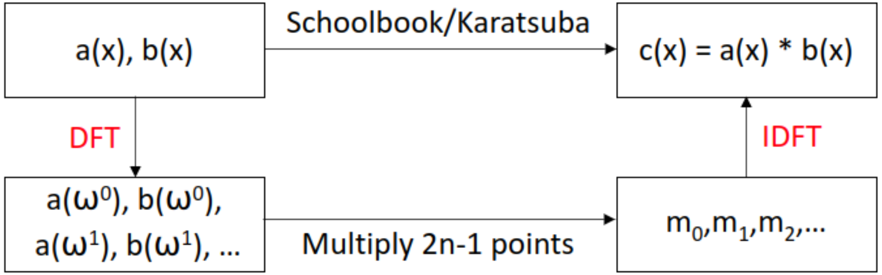
With V(ω) (**DFT**), we compute *evaluation*

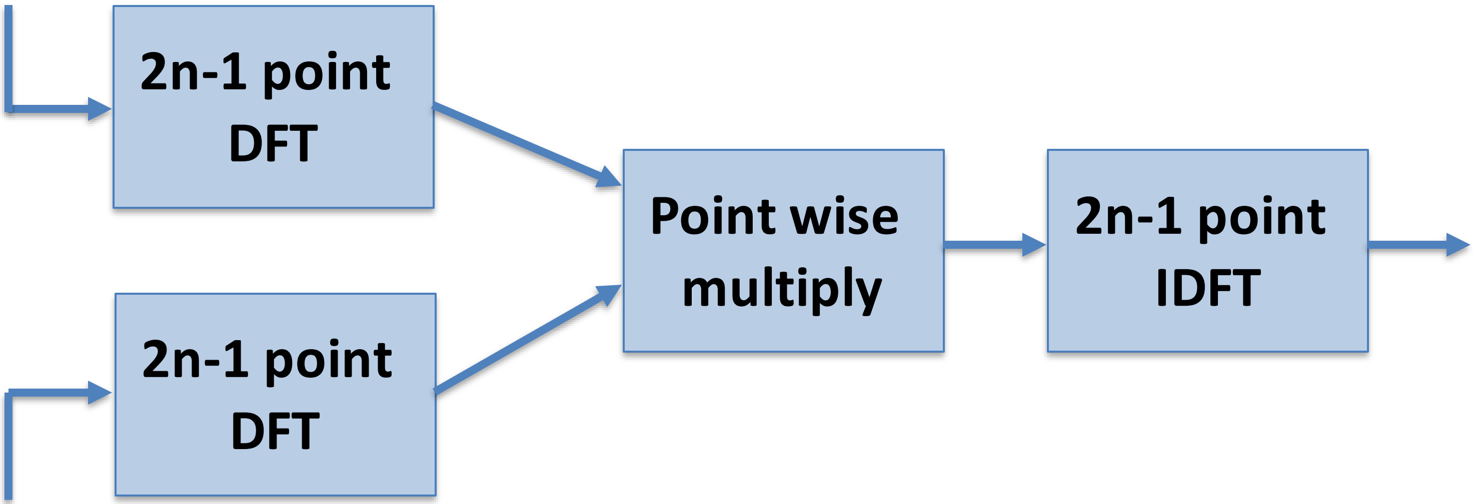
With V(ω)-1 or (1/n) * $V(\omega^{-1})$ (**IDFT**), we compute *interpolation*

- We can use DFT and IDFT for evaluation and interpolation.



Summary: DFT-base polynomial multiplication

$a(x) = a_{n\text{-}1}\, x^{n\text{-}1} + \ldots + a_0$



$b(x) = b_{n\text{-}1}\, x^{n\text{-}1} + \ldots + b_0$

$c(x) = c_{2n\text{-}2}\, x^{2n\text{-}2} + \ldots + c_0$

What is the complexity of Discrete Fourier Transform (DFT) ?

Answer: $O(n^2)$

Fast Fourier Transform (FFT) computes it ‘fast’ in $O(n \log n)$

Fast Fourier Transform (FFT)

The n-point FFT evaluates $a(x) = a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

at n *special* points: $x = \omega_n^k = e^{-i2\pi k/n}$ for $k = 0, \ldots, n-1$ where $\omega_n = e^{-i2\pi/n}$ is the n^{th} primitive root of 1 i.e., $\omega_n^n = 1$.

With these special points, we can **reuse intermediate values** to do fewer computation in total.

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$$\omega_n^{n/2} = -1$$

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We can rewrite

$$\begin{aligned} a(x) &= a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \\ &= (\ldots + a_4 x^4 + a_2 x^2 + a_0) + (\ldots + a_5 x^4 + a_3 x^2 + a_1)x \\ &= a_{even}(x^2) + x a_{odd}(x^2) \end{aligned}$$

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Based on the above,

$$y_k = a(\omega^k) = a_{even}(\omega^{2k}) + \omega^k a_{odd}(\omega^{2k})$$

and

$$y_{k+n/2} = a(\omega^{k+n/2}) = a_{even}(\omega^{2k+n}) + \omega^{k+n/2} a_{odd}(\omega^{2k+n})$$
$$= a_{even}(\omega^{2k}) - \omega^k a_{odd}(\omega^{2k})$$

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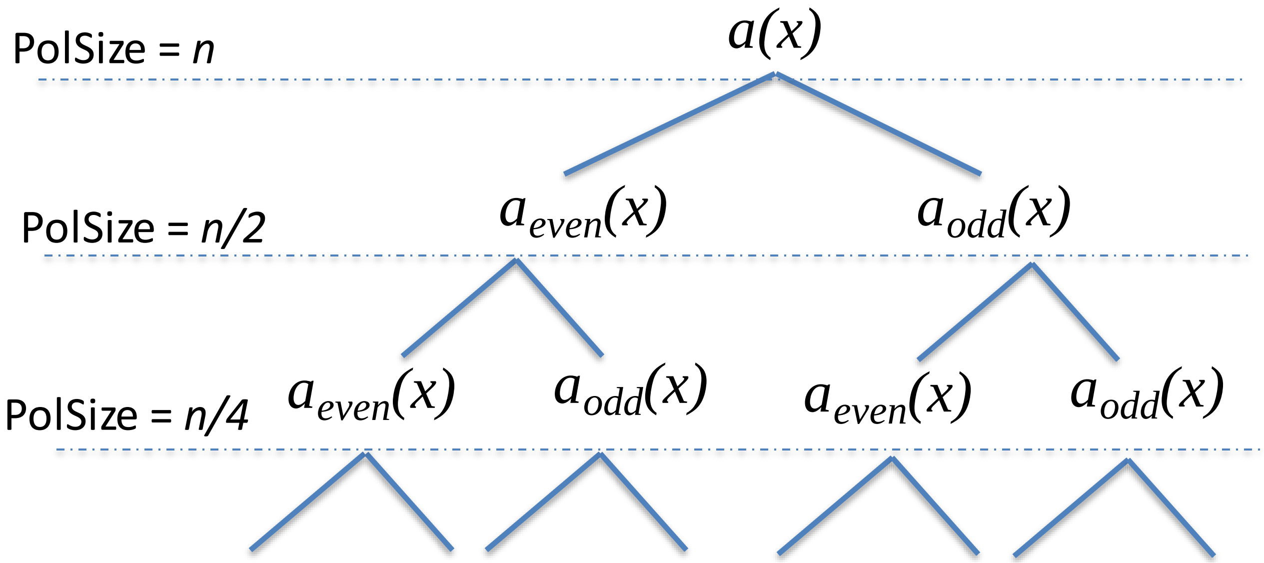
Complexity of FFT

Uses divide and conquer approach



Each level in the tree has O(n) cost. There are log(n) levels.
Total cost = O(n log n)

FFT to Number Theoretic Transform (NTT)

- FFT involves arithmetic of real numbers

  It evaluates at powers of $e^{-i2\pi/n}$ where $e^{-i2\pi/n}$ is the complex n^{th} primitive root of the unity.

- Number Theoretic Transform (NTT)

  NTT replaces $e^{-i2\pi/n}$ by an n^{th} primitive root of the unity modulo q where q is a prime satisfying $q \equiv 1 \bmod n$ and n is a power-of-2.

→ Only ***integer arithmetic*** modulo q

Number Theoretic Transform (NTT)

- An n-point NTT takes a(x) as an input and generates:

$$\mathbf{a}(x) = \sum_{i=0}^{n-1} \mathcal{A}_i . x^i \quad \text{where} \quad \mathcal{A}_i = \sum_{j=0}^{n-1} a_j . \omega^{i.j}$$

ω: n^{th} root of unity (**twiddle factor**) satisfying $\omega^n \equiv 1 (\text{mod } q)$

$\omega^i \neq 1 (\text{mod } q) \; \forall i < n$

$q \equiv 1 \; (\text{mod } n)$

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$q \equiv 1 \; (\text{mod } n)$

- Inverse NTT (INTT) operation uses a similar formula.

$$\text{a}(x) = \sum_{i=0}^{n-1} a_i . x^i \quad \text{where} \quad a_i = \frac{1}{n} . \sum_{j=0}^{n-1} \mathcal{A}_j . \omega^{-i.j}$$

Number Theoretic Transform (NTT)

- Example (NTT for n=4):

$$\mathcal{A}_0 = a_0 + a_1 + a_2 + a_3$$

$$\mathcal{A}_1 = a_0 + a_1.\omega^1 + a_2.\omega^2 + a_3.\omega^3$$

$$\mathcal{A}_2 = a_0 + a_1.\omega^2 + a_2.\omega^4 + a_3.\omega^6$$

$$\mathcal{A}_3 = a_0 + a_1.\omega^3 + a_2.\omega^6 + a_3.\omega^9$$

Using $\omega^4 = 1$

$\omega^2 = -1$

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$$\mathcal{A}_3 = a_0 + a_1.\omega^3 + a_2.\omega^6 + a_3.\omega^9$$

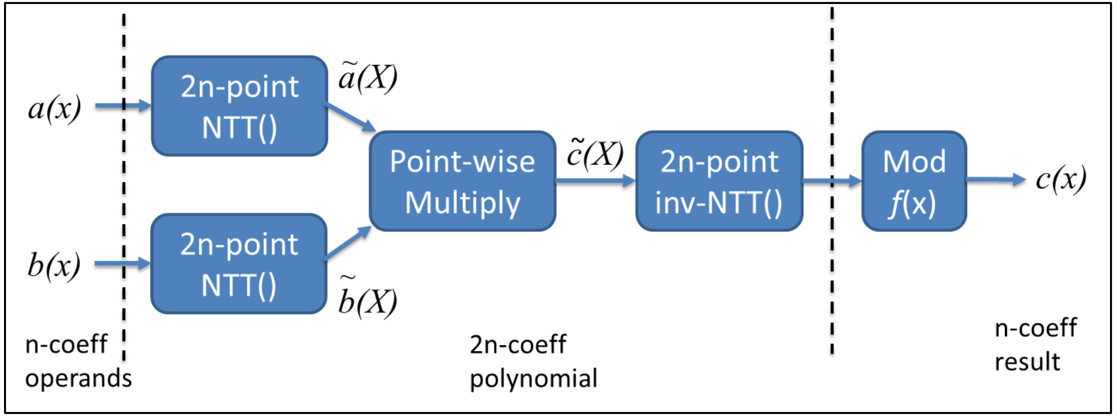
$$\mathcal{A}_0 = a_0 + a_1 + a_2 + a_3$$
$$\mathcal{A}_1 = a_0 + a_1.\omega^1 - a_2 - a_3.\omega^2$$
$$\mathcal{A}_2 = a_0 - a_1 + a_2 - a_3$$
$$\mathcal{A}_3 = a_0 - a_1.\omega - a_2 + a_3.\omega^1$$

Using $\omega^4 = 1$

$\omega^2 = -1$

An optimization in NTT: Negative-wrapped convolution

Polynomial multiplication in $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ where q is a prime satisfying $q \equiv 1 \pmod{n}$ is as follows:



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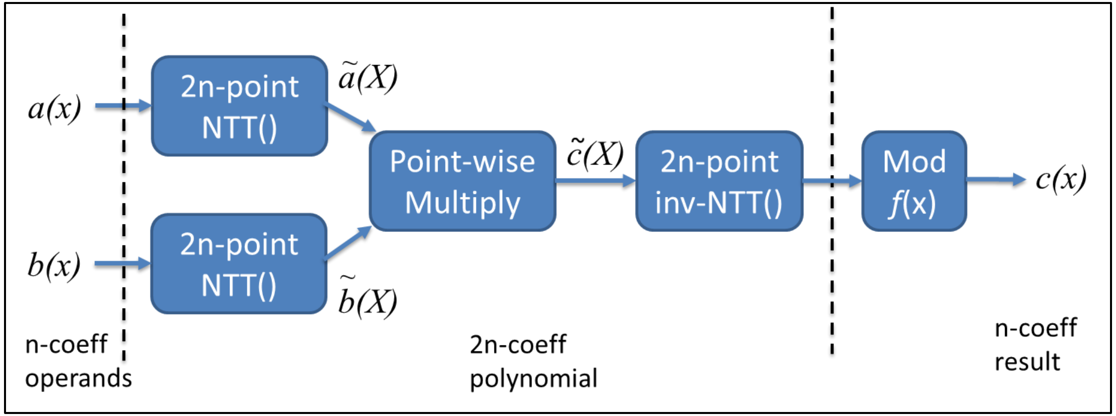
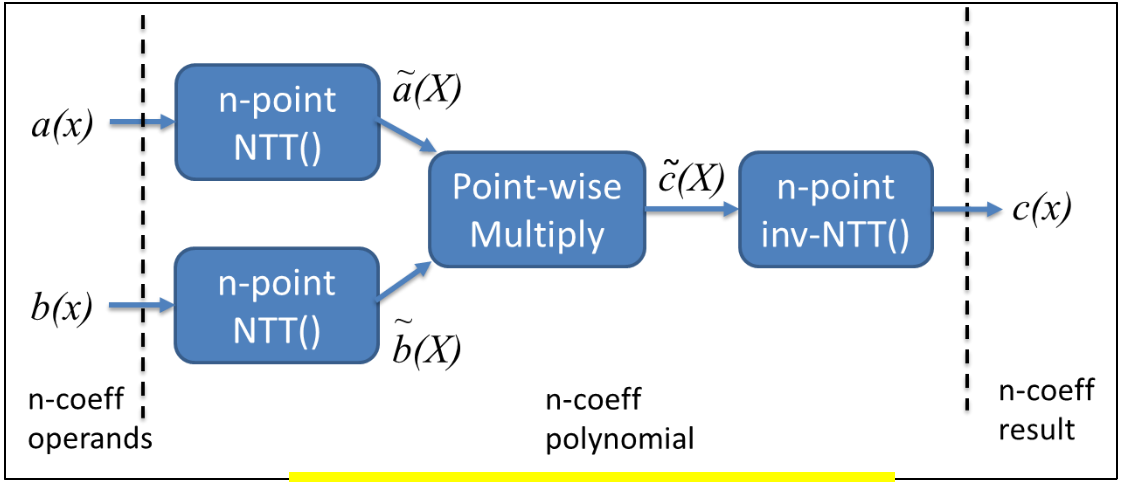
Polynomial multiplication in $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ where q is a prime satisfying $q \equiv 1 \pmod{n}$ is as follows:



Polynomial multiplication in $R_q = \mathbb{Z}_q[x]/\langle f(x) \rangle$ where q is a prime satisfying $q \equiv 1 \pmod{2n}$, and $f(x) = x^n + 1$ is as follows:

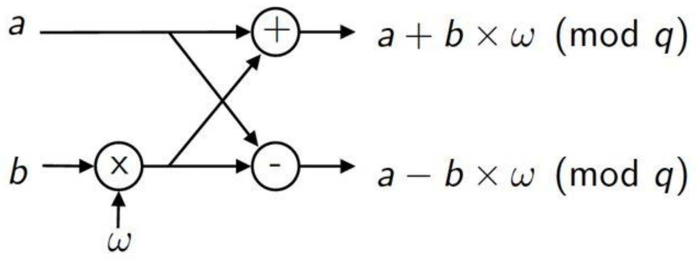


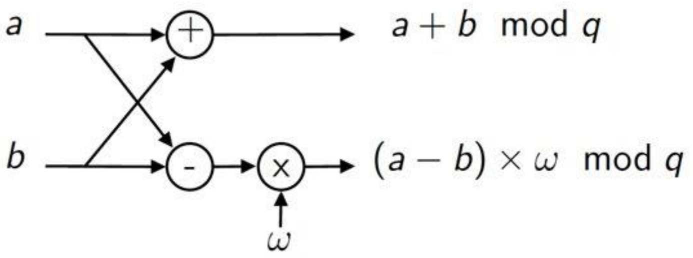
Negative-wrapped convolution

An optimization in NTT: Negative-wrapped convolution

- Two main approaches to perform fast NTT:
  - Decimation-in-time (DIT) with Cooley-Tukey butterfly structure
  - Decimation-in-frequency (DIF) with Gentleman-Sande butterfly structure

- For n-pt NTT, there are log(n) stages where each stage performs n/2 butterfly operations





Explaining NTT using the Chinese Remainder Theorem (CRT)

https://electricdusk.com/ntt.html

(Optional study material. Not essential for this course)

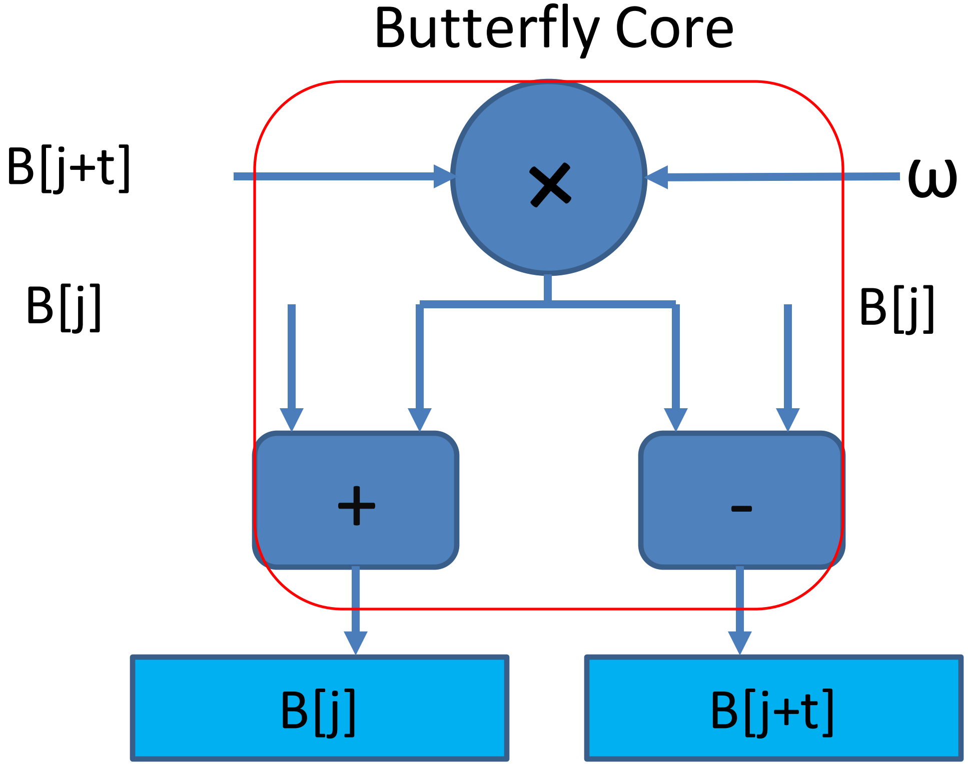
Python code of NTT-based multiplication is available on the course page.

Forward NTT Pseudocode

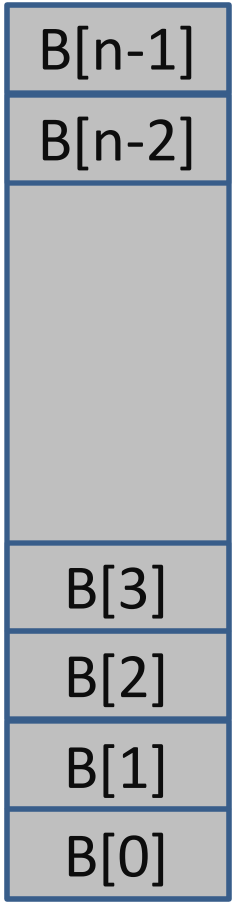
```
fntt(B[ ] of size N):
   t = N
   m = 1
   while(m<N):
       t = int(t/2)
       for i in range(m):
               j1 = 2*i*t
               j2 = j1 + t - 1
               psi_pow = int_bitreverse(m+i)  # Bits in the reverse order

               W = psi_table[psi_pow]

               for j in range(j1,j2+1):              # Cooley-Tukey butterfly operation
                       U = B[j]
                       V = (B[j+t]*W)  % q
                       B[j]   = (U+V) % q
                       B[j+t] = (U-V) % q
       m = 2*m
return B
```

Butterfly circuit for forward NTT

```
# Cooley-Tukey butterfly operation

for j in range(j1,j2+1):
        U = B[j]
        V = (B[j+t]*W) % q
        B[j]   = (U+V) % q
        B[j+t] = (U-V) % q
```



NTT and Memory access



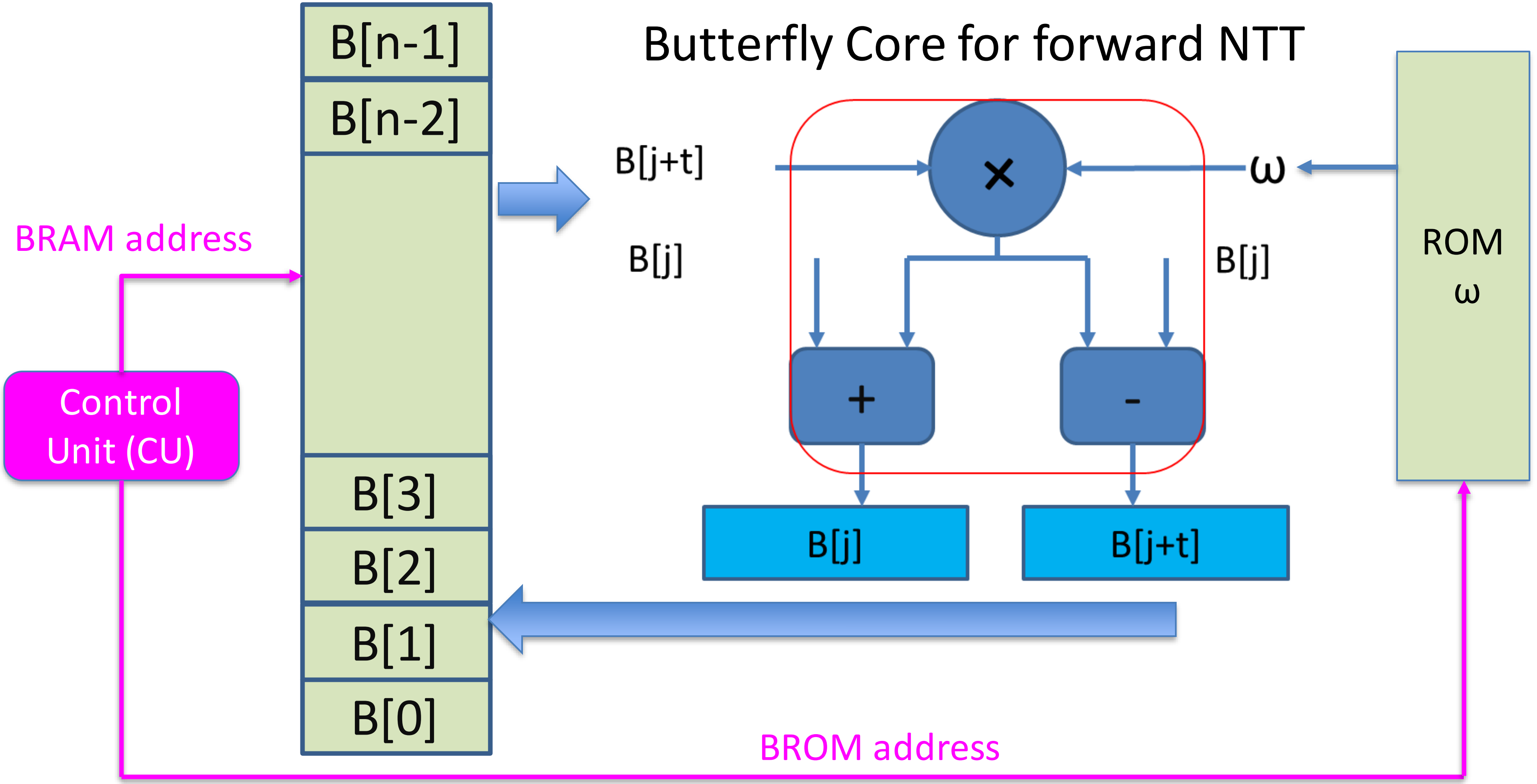
Simplified NTT loops

```
Loop m {
   Loop i {
      Loop j {
            Butterfly(B[j],B[j+t]);
      }
   }
}
```

Butterfly() reads two coefficients from memory.

Butterfly() writes two coefficients to memory.

NTT in HW

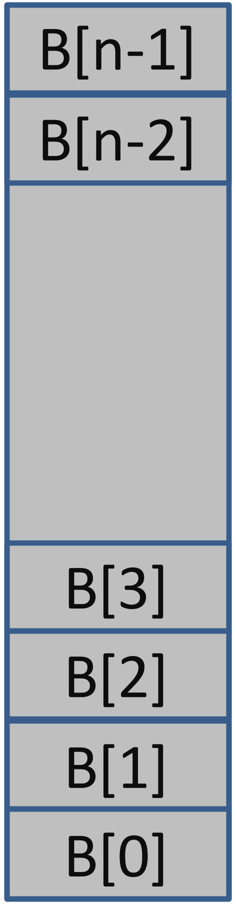


B[n-1]
B[n-2]
B[3]
B[2]
B[1]
B[0]
Butterfly Core for forward NTT
B[j+t]
B[j]
×
ω
B[j]
+
-
B[j]
B[j+t]
ROM
ω
BRAM address
Control Unit (CU)
BROM address

Inverse NTT Pseudocode

```
intt(B[ ] of size N):
    t = 1
    m = N
     while(m>1):
        j1 = 0
        h = int(m/2)
        for i in range(h):
            j2 = j1 + t - 1
            psi_pow = int_bitreverse(h+i,l)
            W = psi_inv_table[psi_pow]

            for j in range(j1,j2+1):
                # Gentleman-Sande butterfly operation
                U = B[j]
                V = B[j+t]
                B[j]   = (U+V) % q
                B[j+t] = (U-V)*W % q
            j1 = j1 + 2*t
        t = 2*t
        m = int(m/2)
        # ... (Division by N)
    return B
```

NTT and Memory access



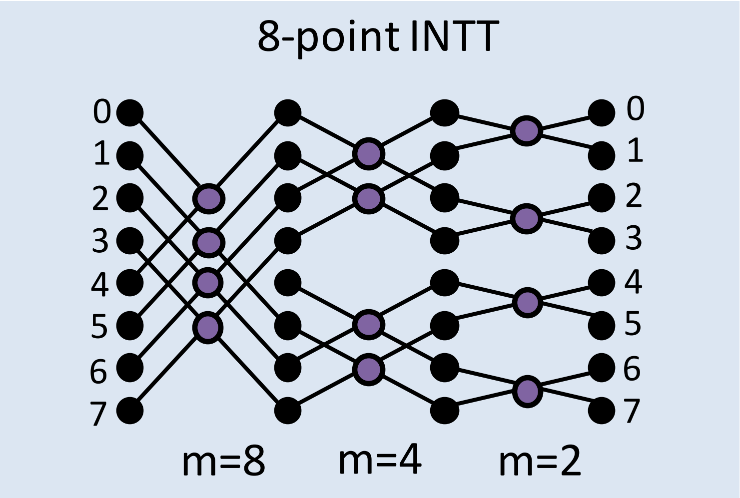
Simplified NTT loops

```
Loop m {
   Loop i {
      Loop j {
            Butterfly(B[j],B[j+m/2]);
      }
   }
}
```

Butterfly() reads two coefficients from memory.

Butterfly() writes two coefficients to memory.

NTT and Memory access

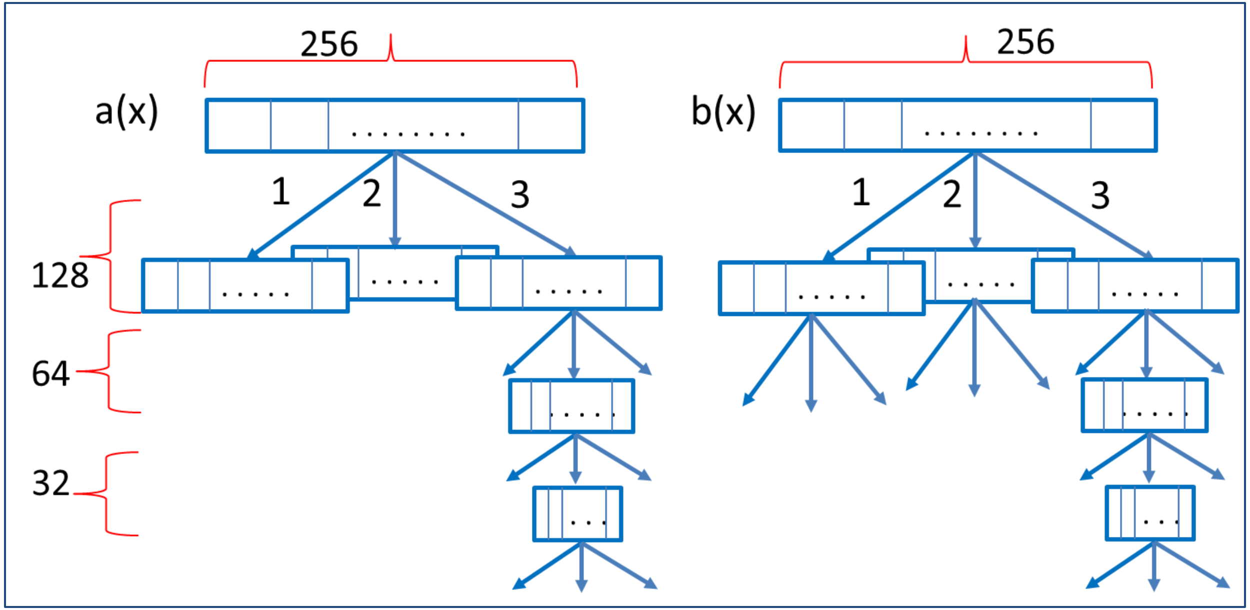


NTT and Memory access

```
--- MFNTT_DIT_NR (N=8)
A_index=0, B_index=4, psi_pow=4
A_index=1, B_index=5, psi_pow=4
A_index=2, B_index=6, psi_pow=4
A_index=3, B_index=7, psi_pow=4
---
A_index=0, B_index=2, psi_pow=2
A_index=1, B_index=3, psi_pow=2
A_index=4, B_index=6, psi_pow=6
A_index=5, B_index=7, psi_pow=6
---
A_index=0, B_index=1, psi_pow=1
A_index=2, B_index=3, psi_pow=5
A_index=4, B_index=5, psi_pow=3
A_index=6, B_index=7, psi_pow=7
```

```
--- MINTT_DIF_RN (N=8)
A_index=0, B_index=1, psi_pow=1
A_index=2, B_index=3, psi_pow=5
A_index=4, B_index=5, psi_pow=3
A_index=6, B_index=7, psi_pow=7
---
A_index=0, B_index=2, psi_pow=2
A_index=1, B_index=3, psi_pow=2
A_index=4, B_index=6, psi_pow=6
A_index=5, B_index=7, psi_pow=6
---
A_index=0, B_index=4, psi_pow=4
A_index=1, B_index=5, psi_pow=4
A_index=2, B_index=6, psi_pow=4
A_index=3, B_index=7, psi_pow=4
```

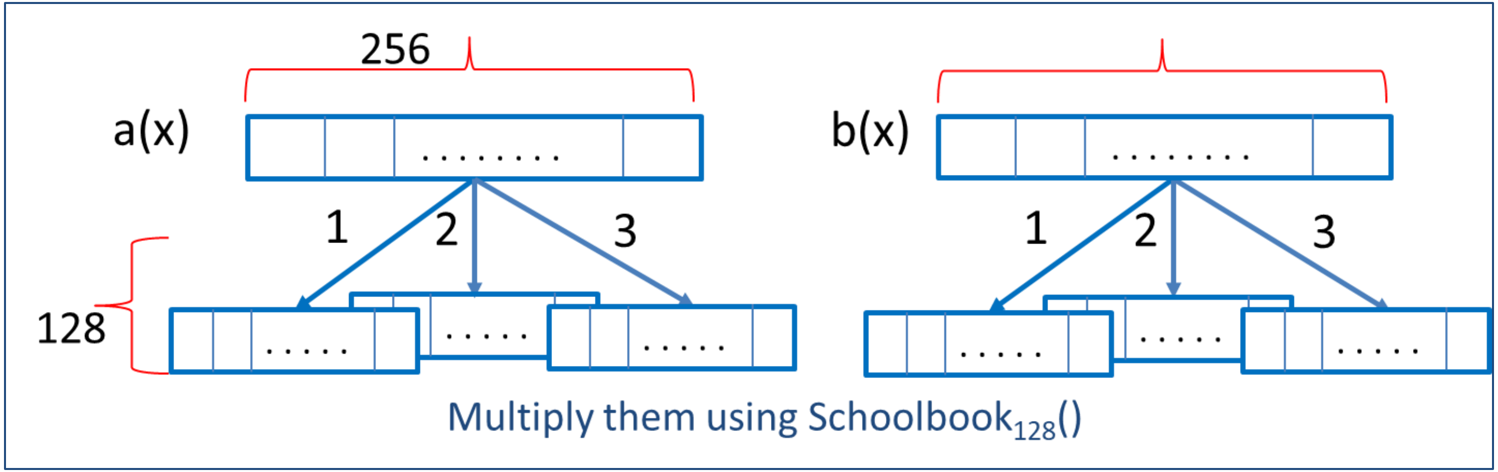
Karatsuba multiplier in HW?



- Karatsuba uses divide-and-conquer recursively.
- Recursion is easy to implement in SW → Call the function recursively.
- Full recursion is '*difficult*' to implement in HW (**my** *personal opinion*)

But, a few levels of recursions is easy to implement. (see next slide)

E.g., 1 level of Karatsuba then Schoolbook

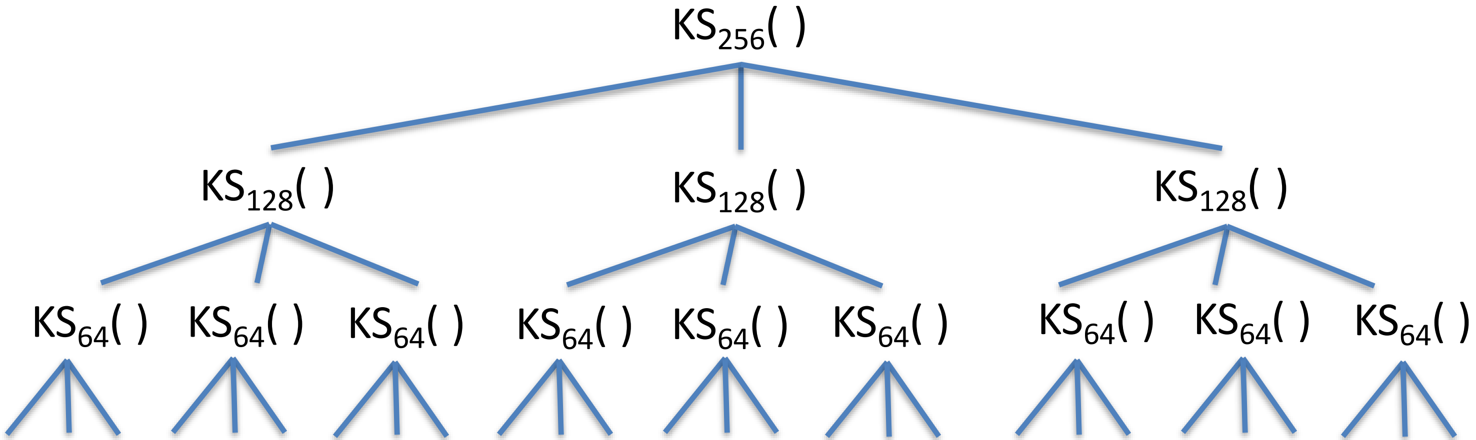


Some ideas:

1. Use HW/SW co-design approach. Perform splitting and joining in SW and compute the Schoolbook multiplications in HW.
   → Easy to implement. But many rounds of HW <--> SW communications.

2. Do everything in HW. → More efficient.

HW/SW co-design of the Karatsuba method



1. SW: Since recursion is challenging to implement in HW, perform all the recursive function calls in SW.

2. HW: When the recursion tree reaches a 'threshold', perform the actual schoolbook multiplications in HW.

3. SW: Read the partial results from HW and combine them in SW.

HW/SW co-design of the Karatsuba method: example