Model Checking with Inductive Invariants

Homework

Find results here:
https://cloud.tugraz.at/index.php/s/zeEgt8ptcRQCXEW

Grades are pseudonymized, your code is the sha256 hash of your matriculation number

Questions? Ask in

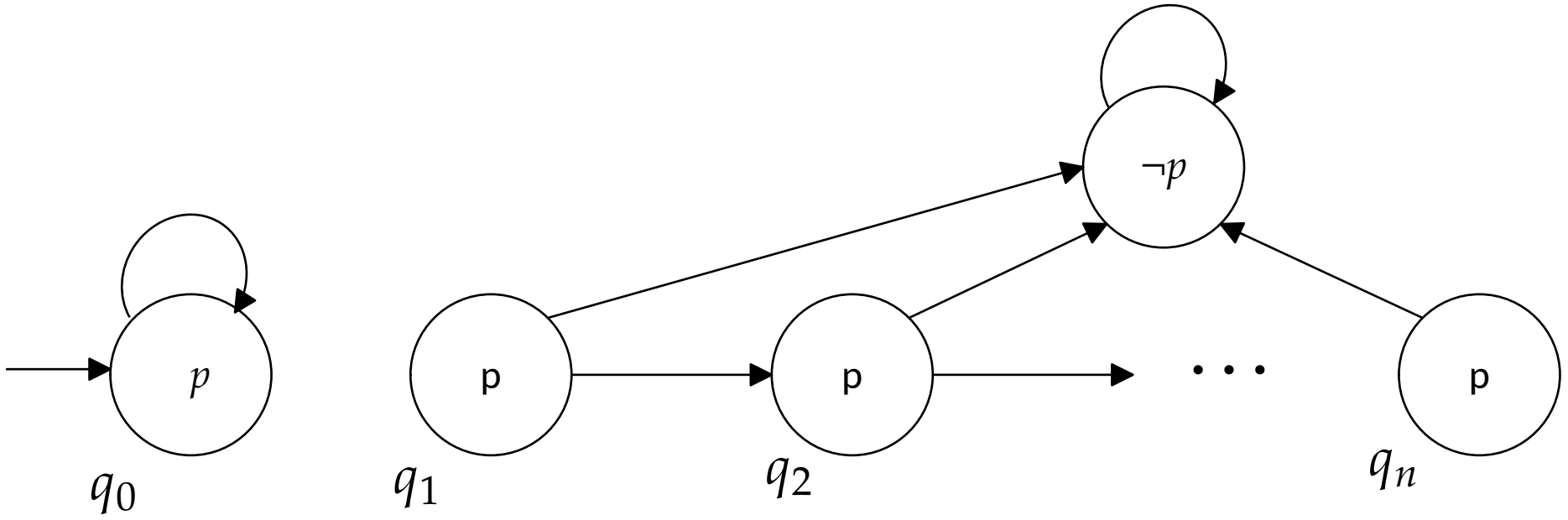
- discord #lecture,
- discord FilipCC, or
- write an email to modelchecking@iaik.

Problems with *k*-induction

Problem: Sometimes *k* is very large

In the following machine, you need $k = n + 1$ to prove $\mathbf{AG}\ p$.

Idea: Automatically find better inductive invariants.

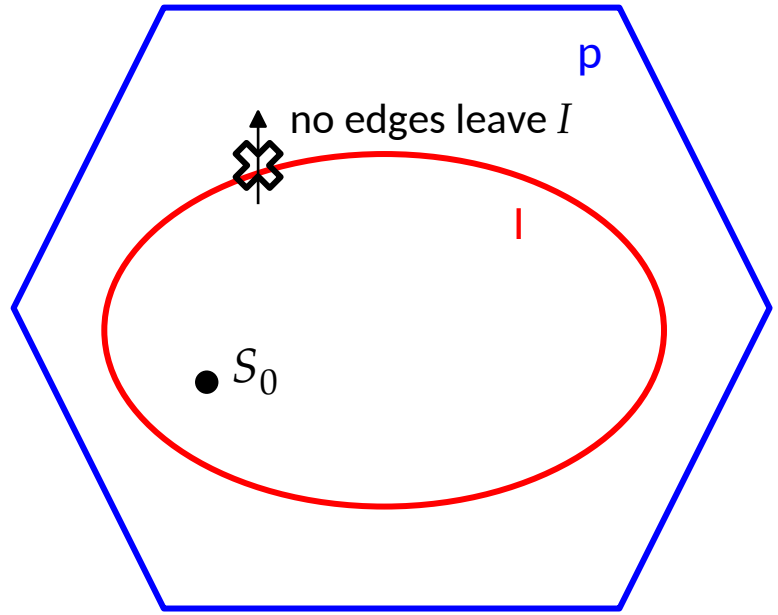


Inductive Invariant

Remember $postimage(Q) = \{ s' \mid \exists s.\ R(s,s') \}$ (see Chapter 5).

Definition. $I \subseteq S$ is an inductive invariant for AG p if

1. $S_0 \subseteq I$
2. $postimage(I) \subseteq I$
3. $\forall s \in I.\ s \vDash p$



If there is an inductive invariant for AG p, then AG p holds.

In formulas:

1. $S_0 \rightarrow I$
2. $I \wedge R \rightarrow I'$
3. $I \rightarrow p$

Inductive Invariant

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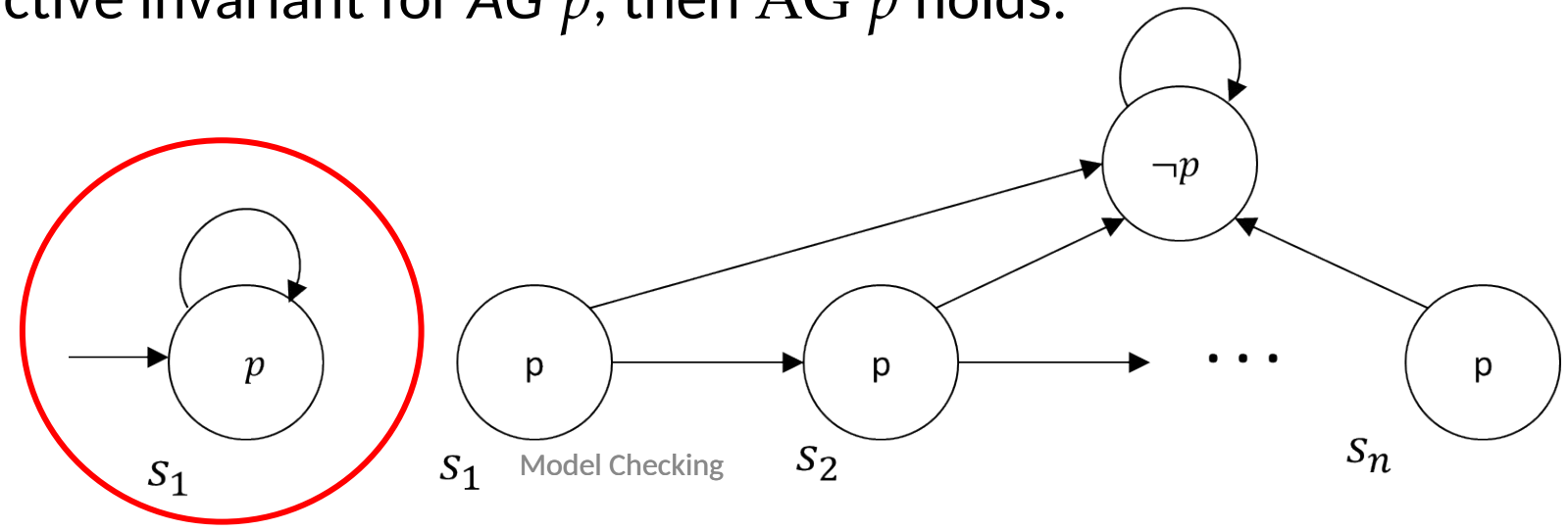
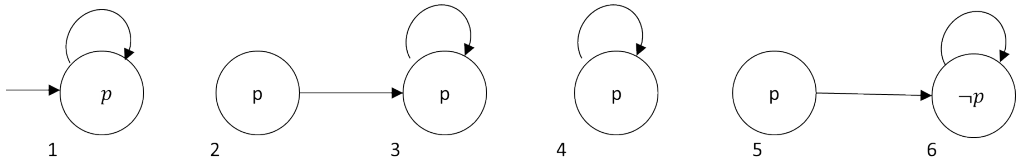
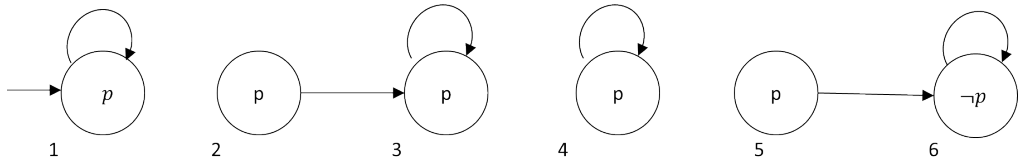
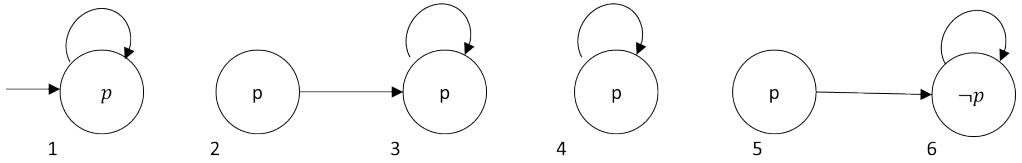
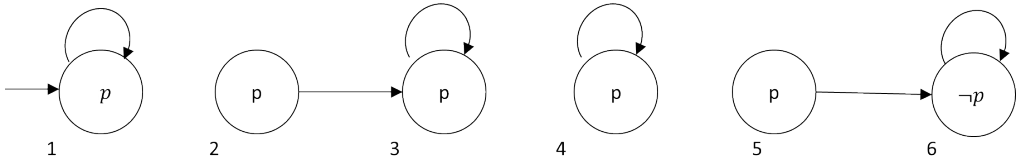


Multiple Invariants

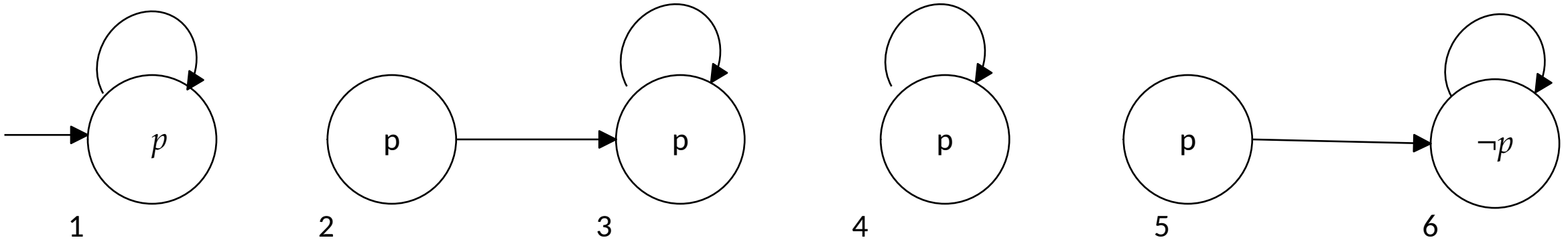








Strongest & Weakest Invariant



Smallest (strongest) invariant is reachable state
Largest (weakest) invariant is states that cannot reach $\neg p$

Model Checking with Craig Interpolants

Ken McMillan, 2003

2010 CAV Award: “has significantly influenced both academic research and industrial practice, and has dramatically changed the scale of systems that can be analyzed by model checking.”

Kenneth McMillan

Interpolants as Inductive Invariants

- BMC finds bugs (and absence of bugs up to k steps)
- How to Show Correctness?
  - k-induction
  - Interpolants

- Find Interpolants I such that
  - States reachable in k steps are in I
  - no bad states are in I
- Interpolants are (good) overapproximation of post-image computation

Interpolant

William Craig, 1957

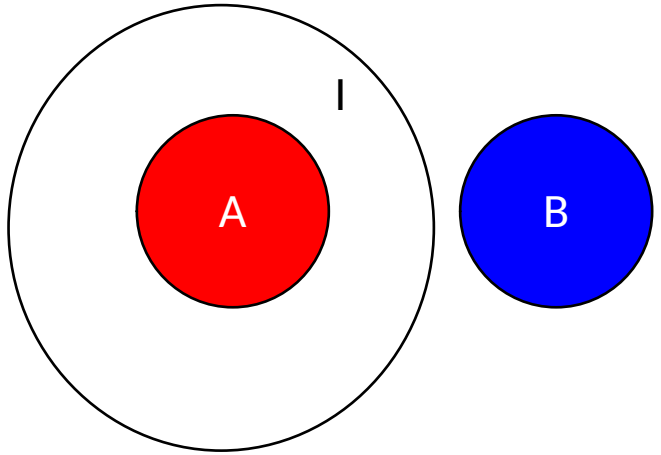
Definition. Given formulas A, B such that $A \wedge B = \bot$, an interpolant is a formula I such that

1. $A \rightarrow I$
2. $I \wedge B \equiv \bot$
3. I only uses symbols that occur both in A and in B

Example. Let

$A = (a_1 \vee \neg a_2) \wedge (\neg a_1 \vee \neg a_3) \wedge a_2$,

$B = (\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4$.



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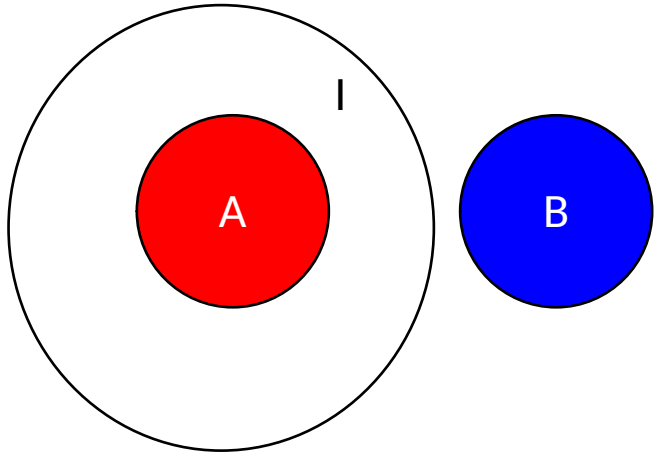


Example. Let

$A = (a_1 \vee \neg a_2) \wedge (\neg a_1 \vee \neg a_3) \wedge a_2$,

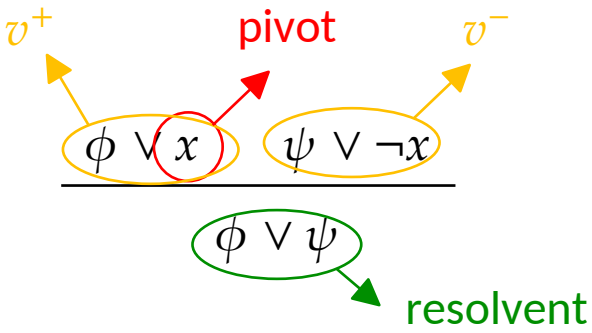
$B = (\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4$.

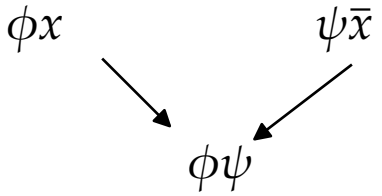
$A \wedge B$ is not satisfiable.

$\neg a_3 \wedge a_2$ is an interpolant:

1. $((a_1 \vee \neg a_2) \wedge (\neg a_1 \vee \neg a_3) \wedge a_2) \rightarrow (\neg a_3 \wedge a_2)$
2. $(\neg a_3 \wedge a_2) \wedge ((\neg a_2 \vee a_3) \wedge (a_2 \vee a_4) \wedge \neg a_4) \equiv \bot$
3. a_2 and a_3 occur in A and in B

Resolution (Chap 9)





Interpolants from Resolution Proofs

For clause C, $C|B$ is obtained by removing literals not in B

Algorithm. Go through resolution proof top-down.

1. If leaf v is labeled $C \in A$, then $Itp(v) = C|B$
2. If leaf v is labeled $C \in B$, then $Itp(v) = \top$
3. If node v has pivot variable $x \in B$ then $Itp(v) = Itp(v^+) \wedge Itp(v^-)$
4. If node v has pivot variable $x \notin B$ then $Itp(v) = Itp(v^+) \vee Itp(v^-)$

Interpolation Example

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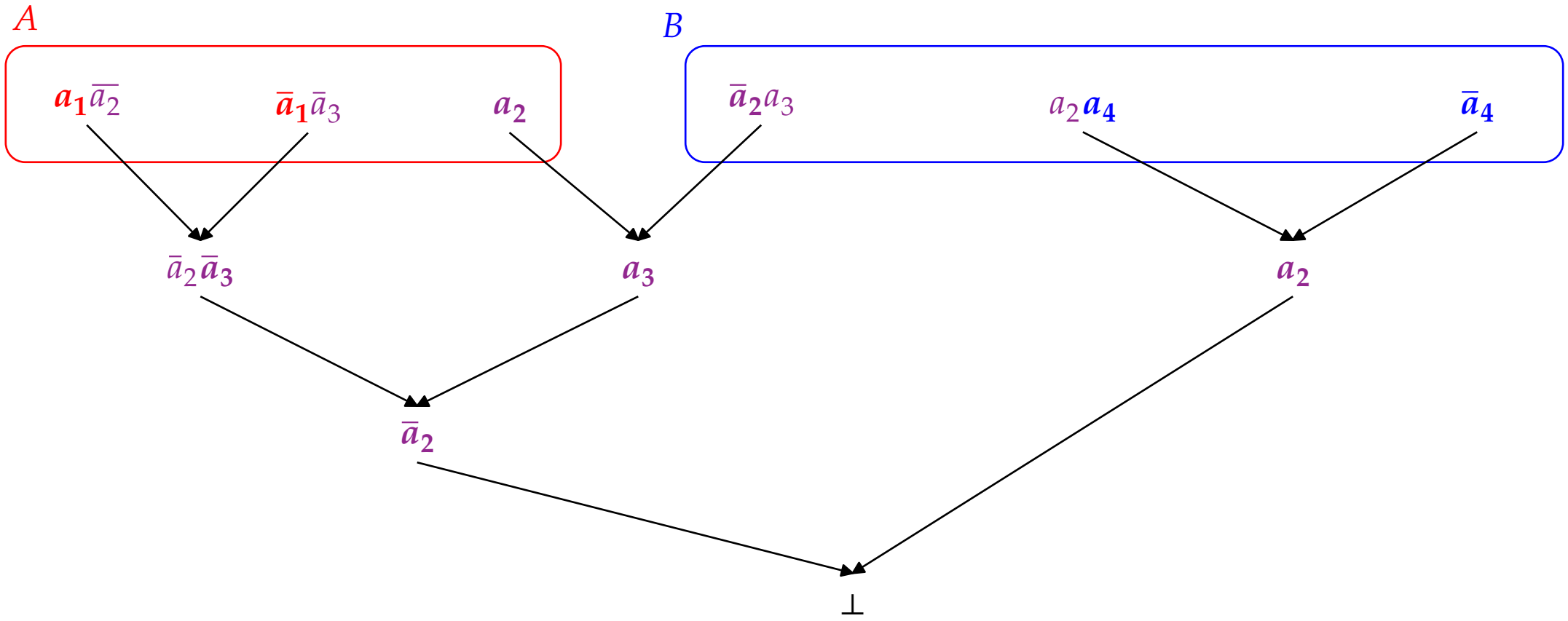

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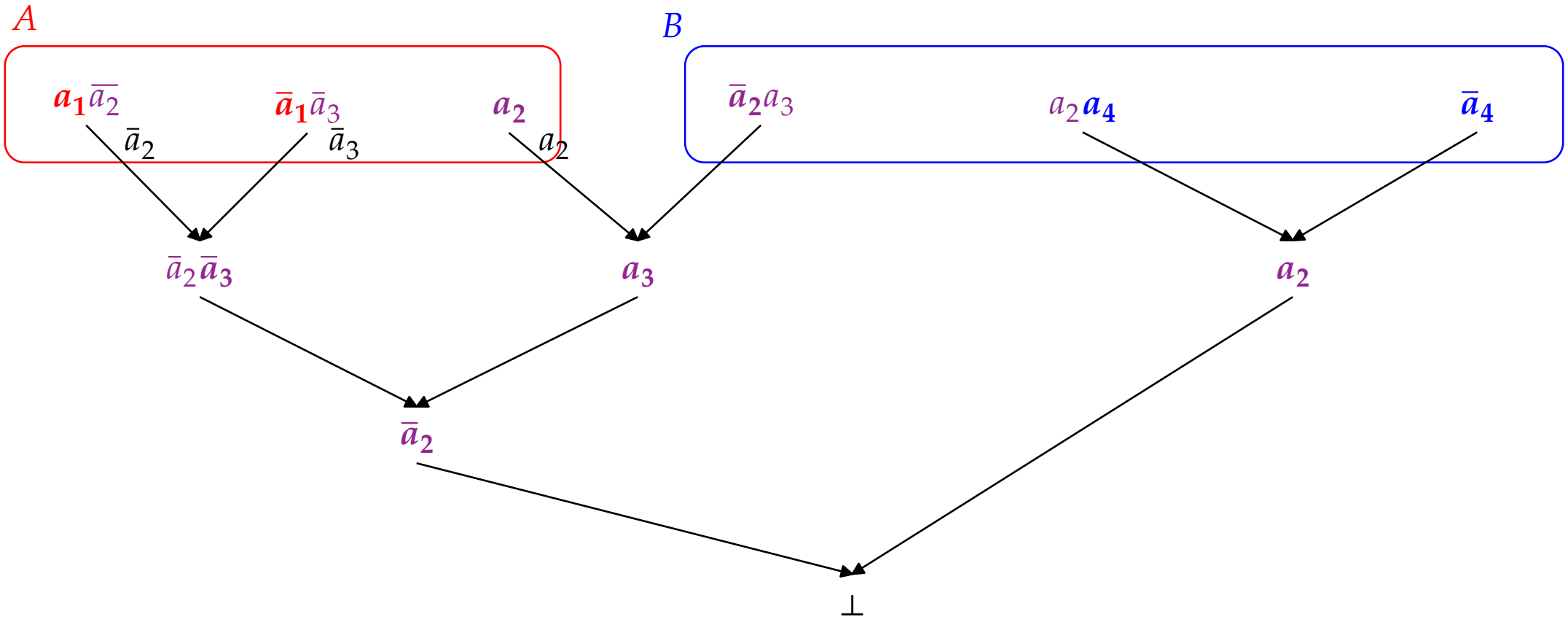

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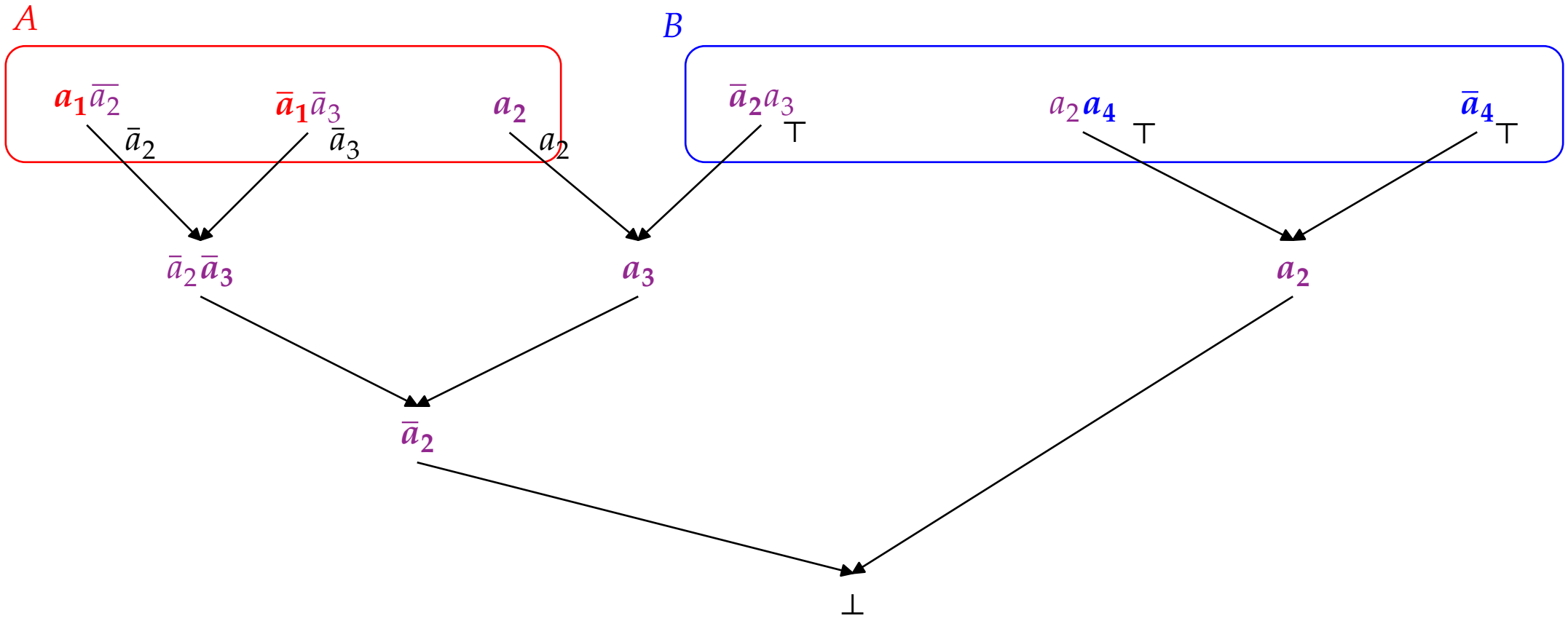

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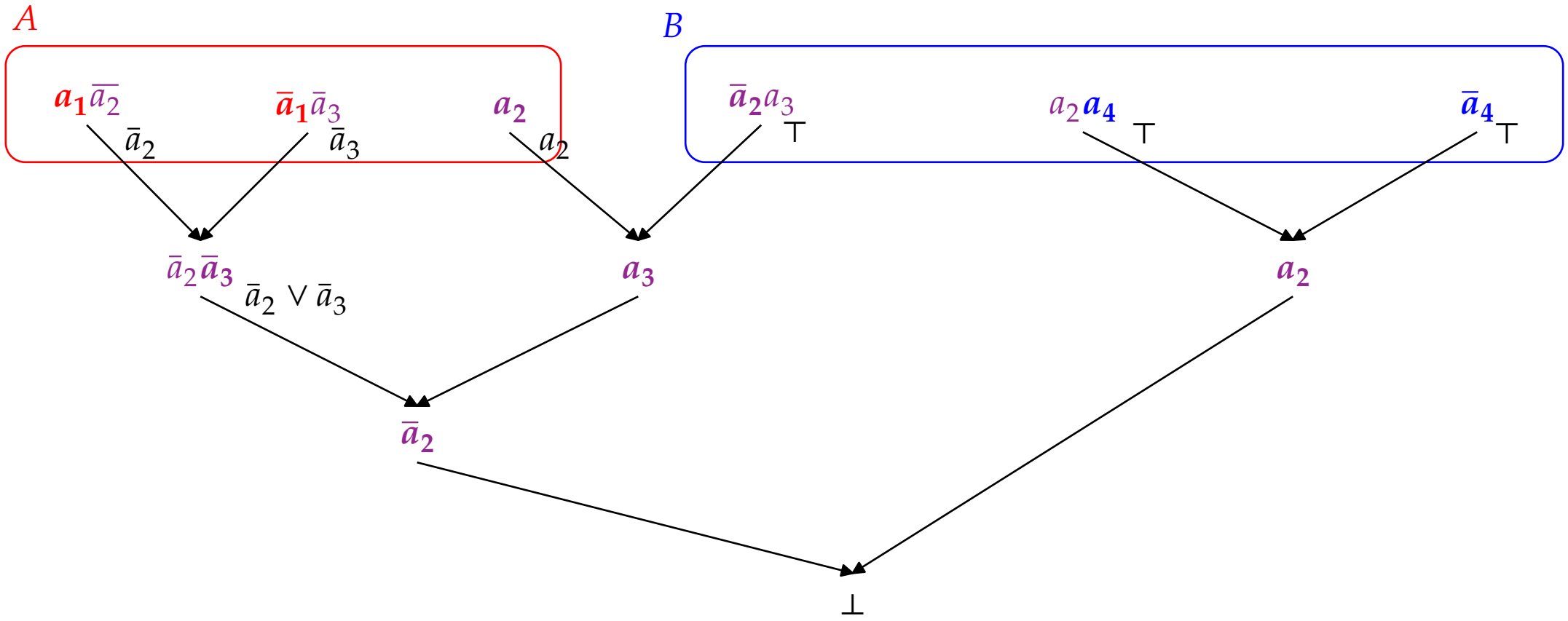

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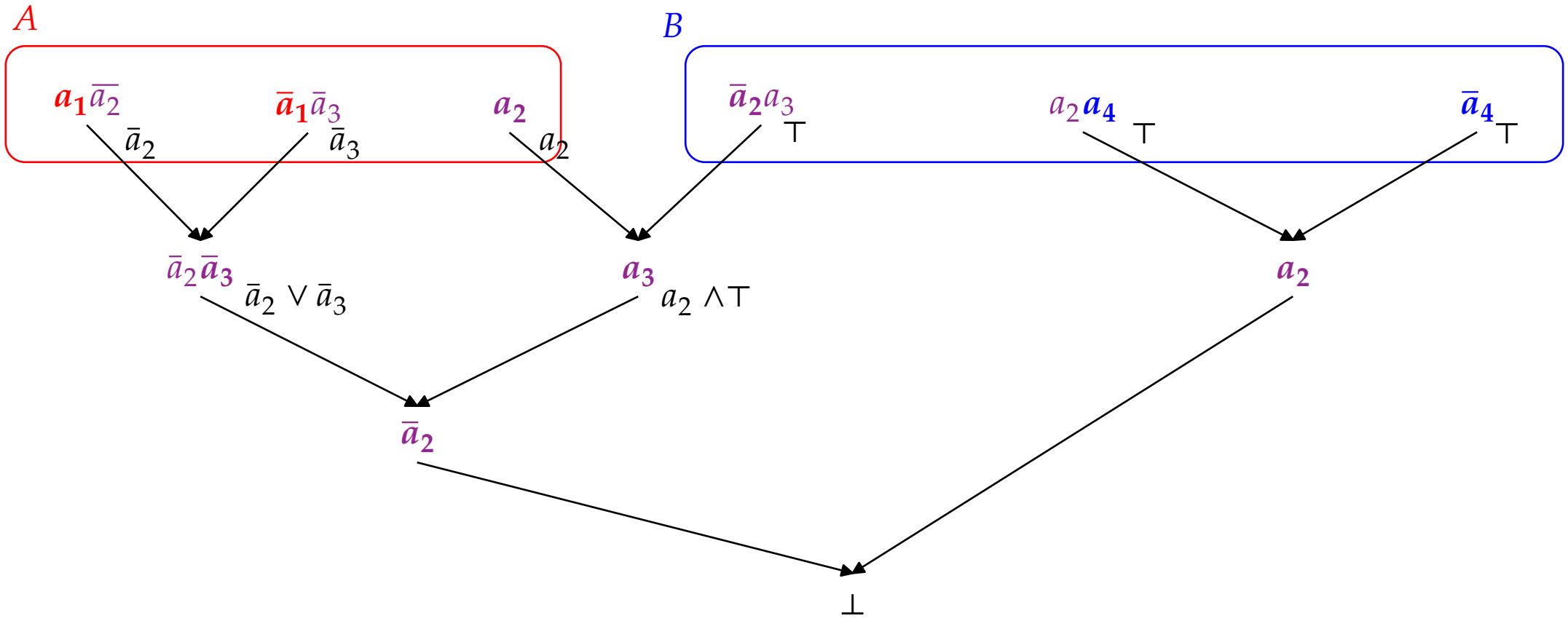

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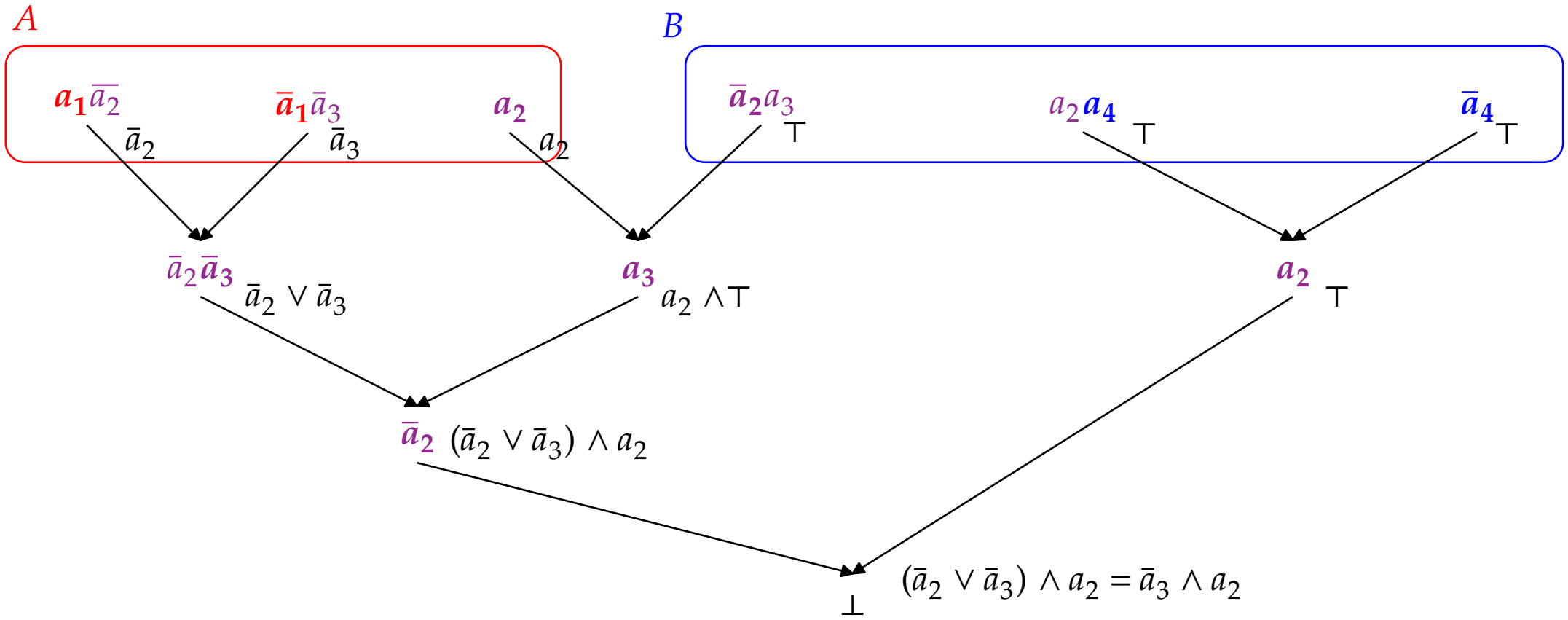


Model Checking with Interpolations

Computation with Overapproximations

BMC to prove p: $S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

Suppose $\mathrm{R} \rightarrow R'$

What does this do?

$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R'(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$

Idea 1:

if $S_0 \wedge \neg$p is SAT return "$M \not\models \mathrm{AG}\, p$ ";

Q := $S_0(s_0)$;

while true **do**

A := $Q(s_0) \wedge R(s_0,s_1)$;

B := $\neg p$;

if $A \wedge B$ is SAT **then**

…

else

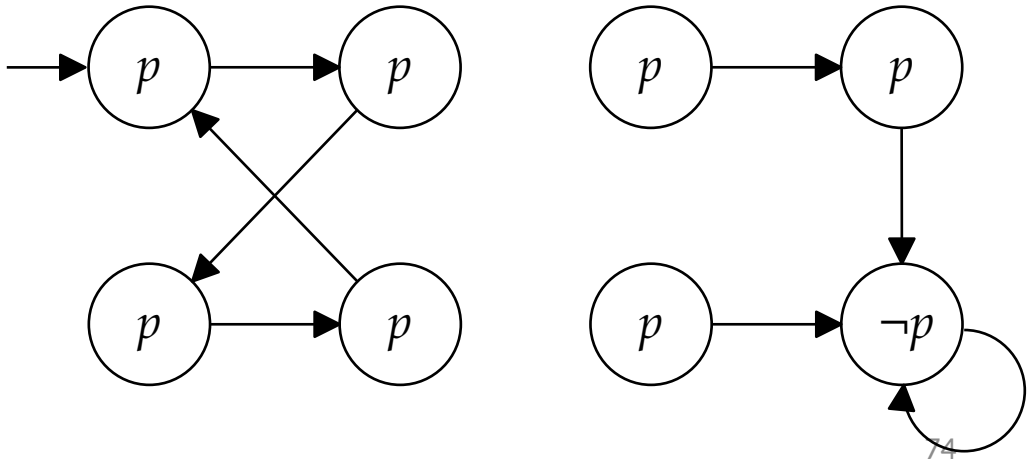
compute interpolant I for A and B;

If $I == Q$ then return "$M \models \mathrm{AG}\, p$";

Q := $Q \vee I$;

end if

end while



What could possibly go wrong?

Reachability Checking with Interpolation

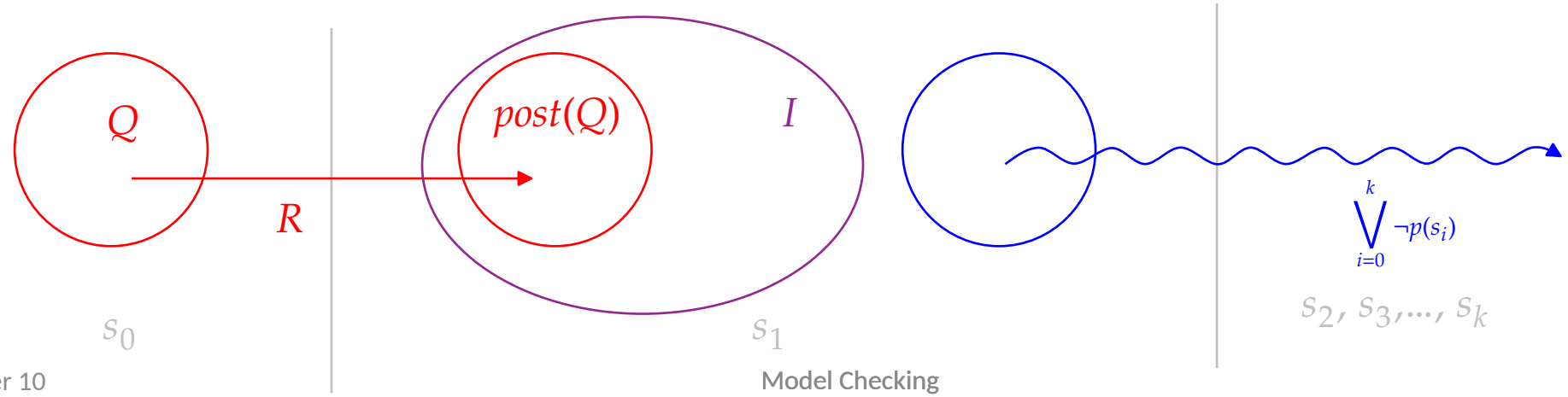
Recall BMC check for $\neg \mathbf{AG} p$:

$S_0(s_0) \wedge \bigwedge_{i=0}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

Instead, start from Q such that $Q \models p$

$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$.

Suppose ϕ unsatisfiable, $I(s_1)$ is an interpolant



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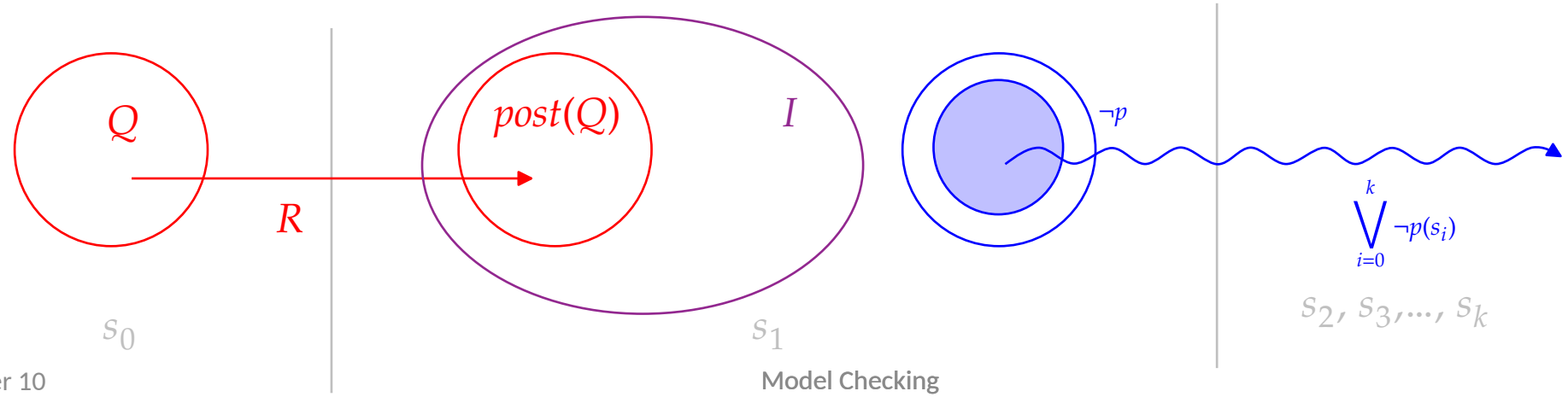
Suppose ϕ unsatisfiable, $I(s_1)$ is an interpolant

Note 1: $\neg p(s_1) \rightarrow B$

so $I(s_1) \wedge \neg p(s_1) = \bot$

Note 2: $I \supseteq post(Q)$



Interpolant Reachability Idea

$\phi = Q(s_0) \wedge R(s_0,s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i,s_{i+1}) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$.

1. Start with $Q = S_0$
2. If ϕ not satisfiable, set Q to $Q \cup I$
3. If Q remains unchanged, p **is not reachable** *(Interpolants are approximation to post-image)*, otherwise goto 2
4. If ϕ is satisfiable and $Q = S_0$, $\neg p$ is **reachable**
5. If ϕ is satisfiable and $Q \neq S_0$, increase k to increase precision of approximation, goto 1.

Procedure terminates when k is diameter of system (or earlier!)

Algorithm

procedure CraigReachability(model M, $p \in AP$)

if $S_0 \wedge \neg$p is SAT return "$M \not\models \mathrm{AG}\ p$ ";

k := 1;

$Q := S_0(s_0)$;

while true **do**

$A := Q(s_0) \wedge R(s_0, s_1)$;

$B := \bigvee_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$;

if $A \wedge B$ is SAT **then** // $\neg p$ can be reached from Q

if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$"; // $\neg p$ can be reached from S_0

Increase k // Not sure if path to $\neg p$ is real. Increase precision

$Q := S_0(s_0)$;

else

compute interpolant I for A and B;

If $I(s_0) == Q$ then return "$M \models \mathrm{AG}\ p$"; // Reached the fixpoint of overapproximated reachability?

$Q := Q \vee I(s_0)$; // Another step of overapproximated reachability?

end if

end while

if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

  Q := $S_0(s_0)$;

else

  compute interpolant I for A and B;

  if $I(s_0) \rightarrow Q$ then return "$M \models \mathrm{AG}\ p$";

  Q:= $Q \vee I(s_0)$;

end if

10.4.4 Correctness

If CraigReachability returns "$M \models AG\ p$" then $M \models AG\ p$

Let Q_i denote Q at iteration i. For all i, $Q_i \leftarrow postimage^i(Q_0)$. If $I \rightarrow Q_i$, we have reached a fixed point $Q^* = Q_i$ so $Q^* \leftarrow postimage^*(Q_0)$. Now because $Q_i \wedge \neg p = \bot$, we have $postimage^*(Q_0) \wedge \neg p = \bot$.

If CraigReachability returns "$M \not\models AG\ p$" then $M \not\models AG\ p$

$A \wedge B$ encodes a path from Q_0 to $\neg p$.

CraigReachability terminates

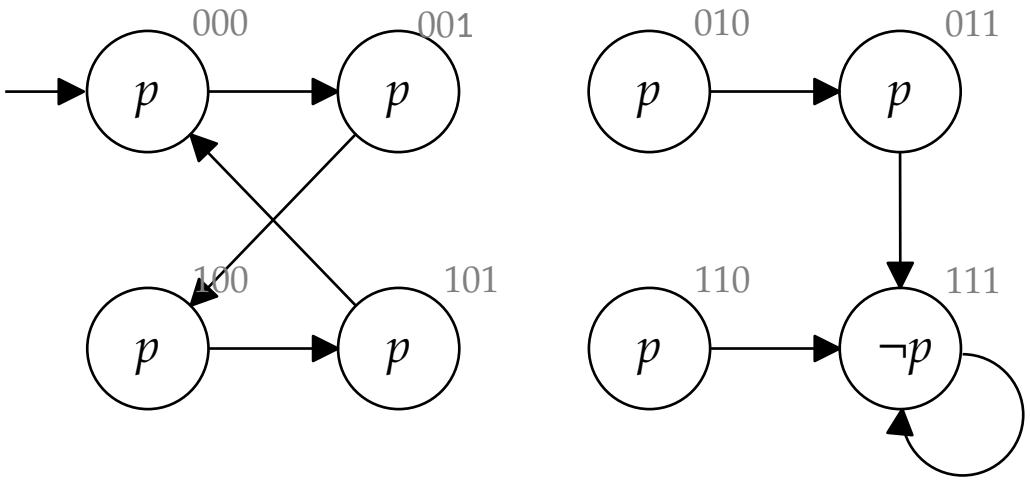
Note that k increases.

If $M \not\models AG\ p$, there is a path of length l to $\neg p$ and we will find it when $l = k$.

Suppose $M \models AG\ p$. If k is the diameter of the graph, no I and thus no Q_i can contain a state that reaches $\neg p$. Thus, $A \wedge B$ is never SAT and the algorithm terminates because the Q_i cannot grow forever.

$x_1 x_2 x_3$

Example **AG** p



$$\phi = Q(s_0) \wedge R(s_0, s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i, s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i).$$

if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

 Q := $S_0(s_0)$;

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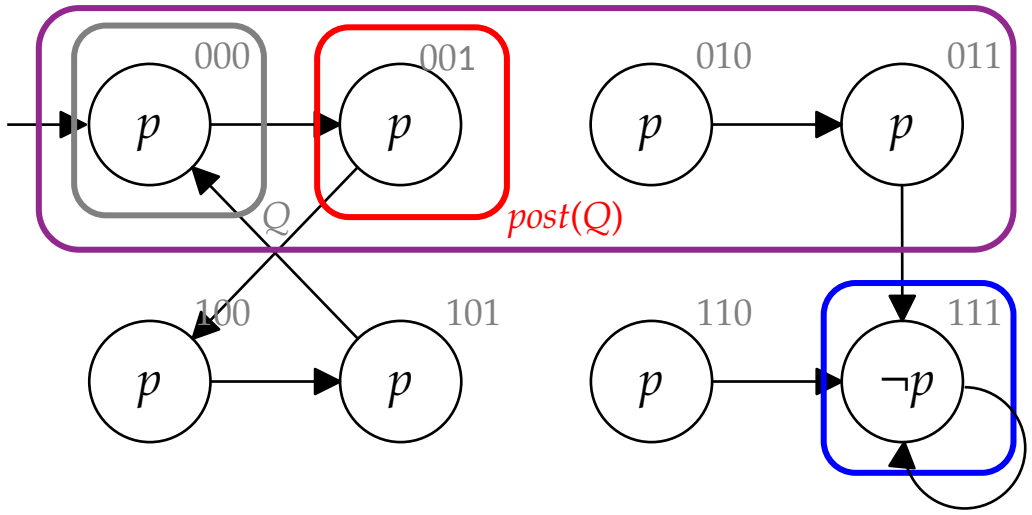
Example $\mathbf{AG}\ p$



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$\boldsymbol{k = 1.}$

$Q = \neg x_1 \wedge \neg x_2 \wedge \neg x_3 = \{000\}.$

ϕ is UNSAT

Invariant checks first bit: $I = \neg x_1$

if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

 Q := $S_0(s_0)$;

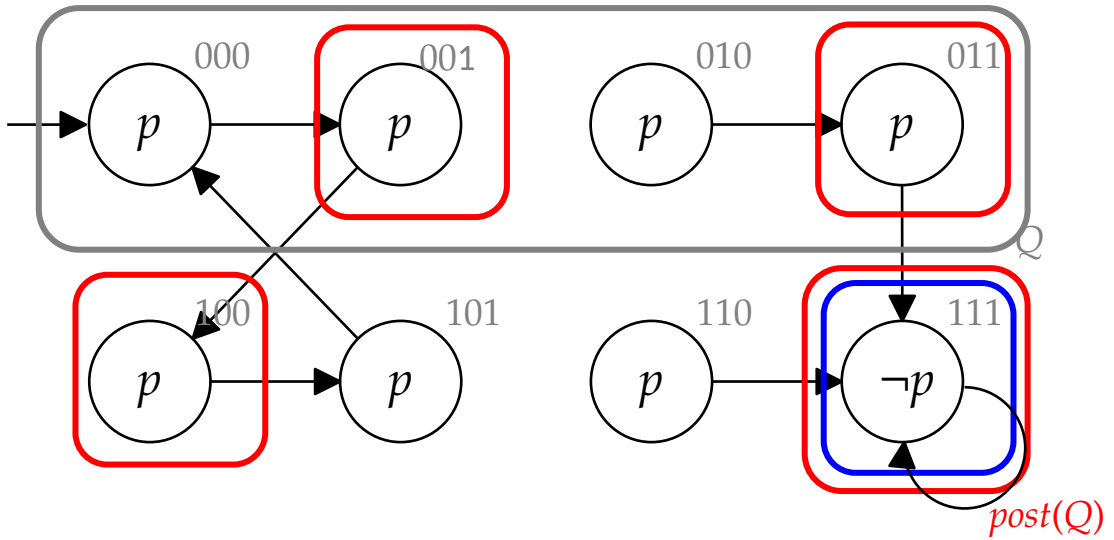
else

  compute interpolant I for A and B;

  if $I(s_0) \rightarrow Q$ then return "$M \models \mathrm{AG}\ p$";

 Q:= $Q \vee I(s_0)$;

Example **AG** p



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$k = 1.$

$Q = \neg x_1 = \{000,001,010,011\}.$

ϕ is SAT

if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

 $Q := S_0(s_0)$;

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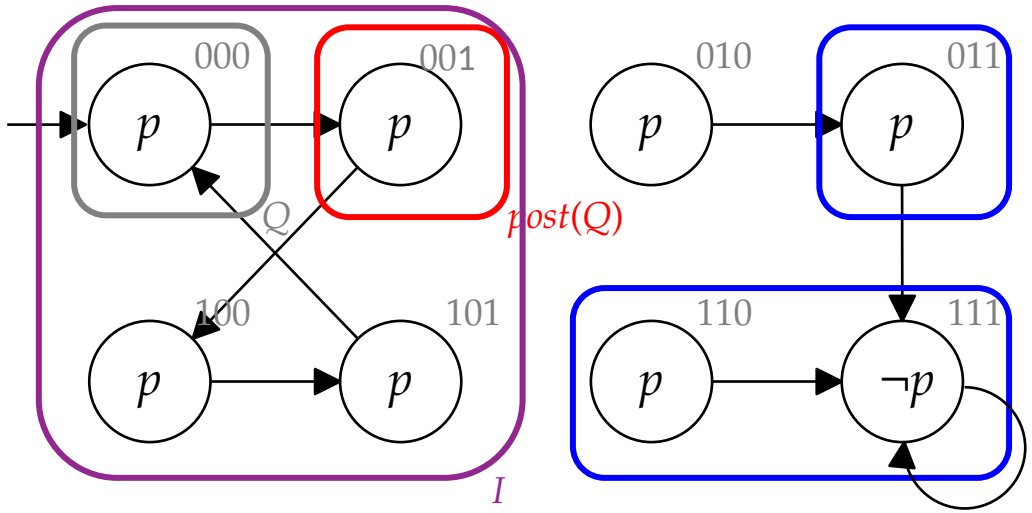
 $Q := Q \vee I(s_0)$;

Example **AG** p



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$\boldsymbol{k = 2.}$

$Q = \neg x_1 \wedge \neg x_2 \wedge \neg x_3 = \{000\}.$

ϕ is UNSAT

Invariant checks 2nd bit: $I = \neg x_2$

if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

  $Q := S_0(s_0)$;

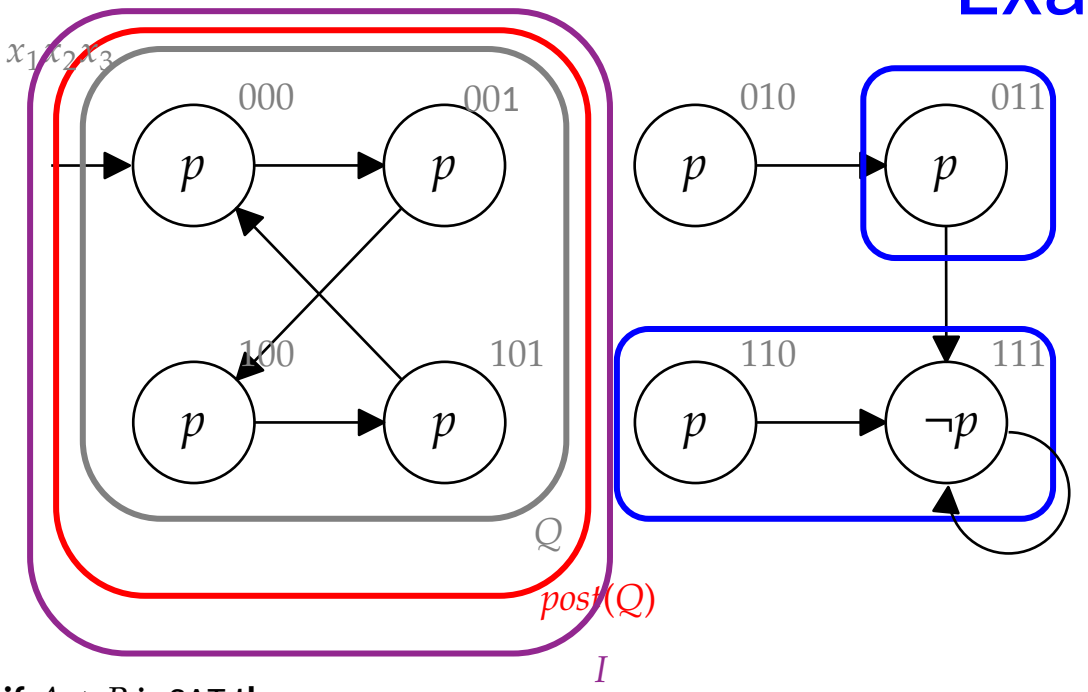
else

  compute interpolant I for A and B;

  if $I(s_0) \rightarrow Q$ then return "$M \models \mathrm{AG}\ p$";

  $Q := Q \vee I(s_0)$;

Example $\mathbf{AG}\ p$



if $A \wedge B$ is SAT **then**

  if $Q = S_0$ then return "$M \not\models \mathrm{AG}\ p$";

  increase k

  $Q := S_0(s_0)$;

else

  compute interpolant I for A and B;

  if $I(s_0) \rightarrow Q$ then return "$M \models \mathrm{AG}\ p$";

  $Q := Q \vee I(s_0)$;

$\phi = Q(s_0) \wedge R(s_0,s_1) \wedge \bigwedge_{i=1}^{k-1} R(s_i,s_{i+1}) \wedge \bigvee_{i=1}^{k} \neg p(s_i)$.

$k = 2$.

$Q = \neg x_2 = \{000, 001, 100, 101\}$

ϕ is UNSAT

$I = \neg x_2 = Q$.

Algorithm terminates.

How Did I Pick the Interpolants?

What I did

- Start with $A = postimg(Q)$
- Perform each of the following steps
  1. Can I throw away x_3? (Is $(\exists x_3.A) \cap B = \varnothing$?) If yes, $A := \exists x_3.A$
  2. Can I throw away x_2? If yes, $A := \exists x_2.A$
  3. Can I throw away x_1? If yes, $A := \exists x_1.A$

This hack only works because the postimg(Q) is a state or a cube!

In the homework, you will get CNFs. Try getting rid of clauses.

Note that A is always a valid interpolant.

Next Week

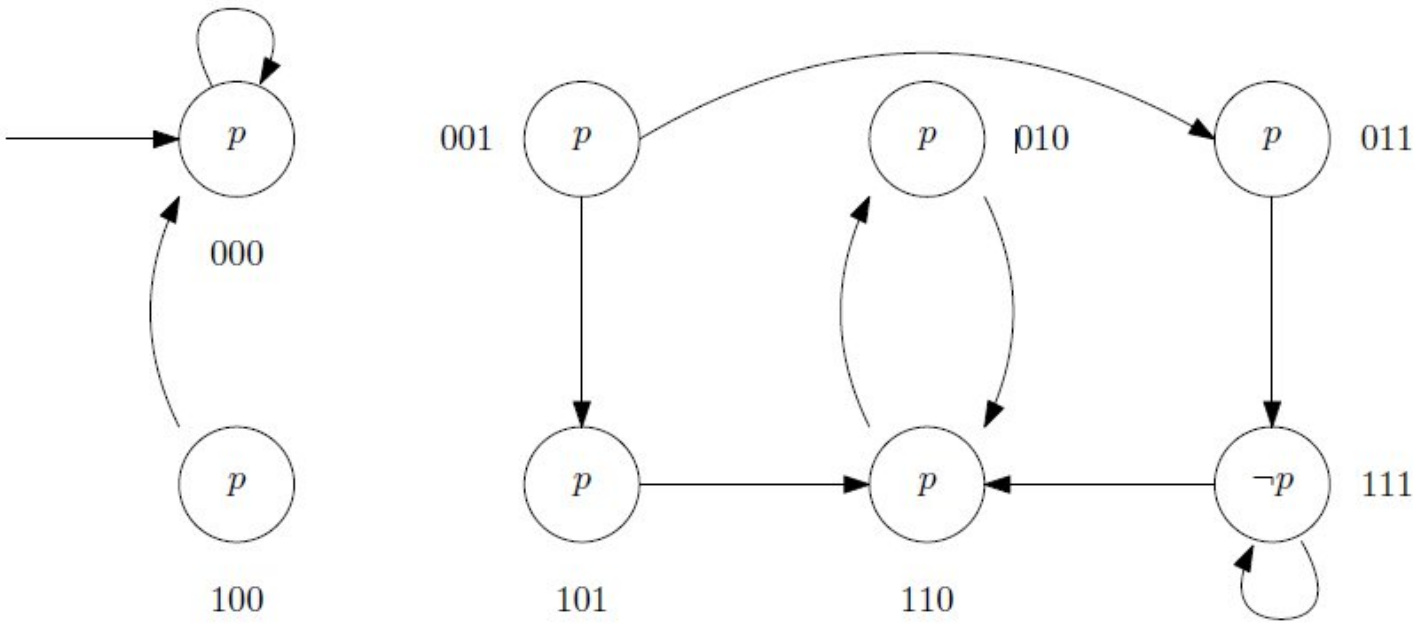
2PM i13: Property-Directed Reachability

Homework!

Model Checking (SS 2023) Homework 3

Deadline: April 13, **2023, 4:00 pm**
Send your solution to `modelchecking@iaik.tugraz.at`
Homework can be done in groups of 1 or 2 students.
The groups need not be the same for each homework.
Indicate clearly which students present the homework.

Consider the following Kripke structure K, with states $(x_1, x_2, x_3) \in \{0,1\}^3$ and atomic proposition p.



Task 1. [**10 points**] Use Model Checking with Craig Interpolants to prove whether the property $AG\, p$ is true or false.

Clearly indicate the steps. I would like to see the interpolants as formulas, for anything else, you can use set notation. You can also draw the sets, but use enough copies of the Kripke structure to make sure we can understand your steps, at least one for every k.

Use the same heuristic shown in class to find the interpolants. The heuristic shown in class is a hack, but it works in this example.