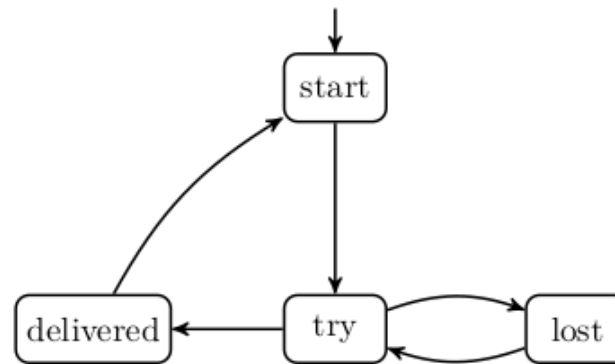


Probabilistic Model Checking

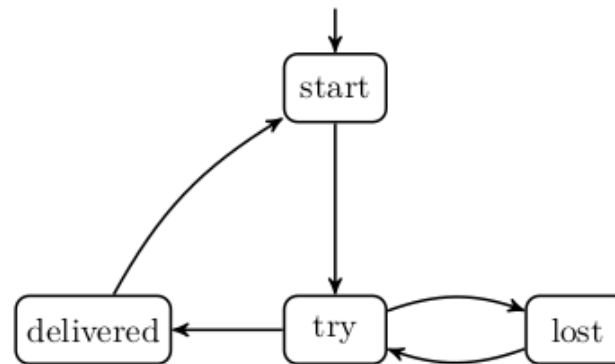
Stefan Pranger

01. 06. 2023

Communication Protocol



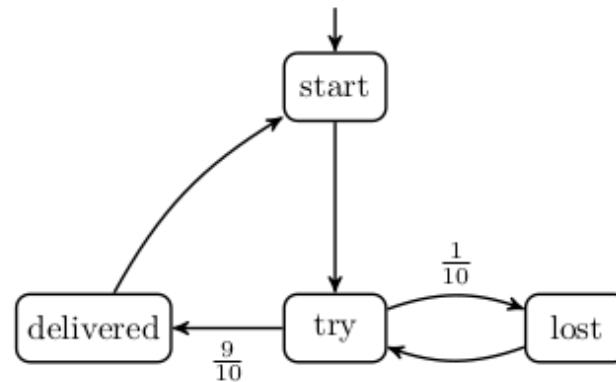
Communication Protocol



But $\mathcal{M}, start \models \exists \mathbf{G} \neg delivered$?

or $\mathcal{M}, start \models \forall \mathbf{F} delivered$?

Communication Protocol



But $\mathcal{M}, start \models \exists \mathbf{G} \neg delivered$?

or $\mathcal{M}, start \models \forall \mathbf{F} delivered$?

Does not make sense with probabilities! \rightarrow We *need* new descriptions for properties.

We have different models.

Markov Chains

Markov Chain $\mathcal{M} = (S, \mathbb{P}, s_0, AP, L)$

- S a set of states and initial state s_0 ,
- $\mathbb{P} : S \times S \rightarrow [0, 1]$, s.t.

$$\sum_{s' \in S} \mathbb{P}(s, s') = 1 \quad \forall s \in S$$

- AP set of atomic propositions and $L : S \rightarrow 2^{AP}$ a labelling function.

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- What is the probability to reach the destination without ever running into an unsafe area?
- What is the probability to send 6 messages successfully and only failing a maximum amount of 15 times?

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- Describe states through variables:
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 - *agent_is_on_slippery, ...*
 - ...

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- Describe states through variables:
 - $x \in [0, 20], y \in [0, 20], velocity \in [0, 1], \dots$
 - *processor_one_idle, processor_two_idle, \dots*
 - *agent_is_on_slippery, \dots*
 - ...
- For each possible state we describe the possible variable updates:
 - If $x > 10$ & $y < 10$ & *agent_is_on_slippery* then the agent moves to one of its adjacent cells each with probability $1/4$.
 - If *processor_one_idle* & *processor_two_idle* then the process will be processed by processor one or two.

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 - If *processor_one_idle* & *processor_two_idle* then the process will be processed by processor one or two.
 - If *processor_one_idle* & *processor_two_idle* then we can **decide** to use processor one or two.

The PRISM Modelling Language

- Modules: Group associated behaviour

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module processor1 ... endmodule  
module processor2 ... endmodule
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b : bool init false;  
global temperature : [0..100] init 32;  
const double pi = 3.14;
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[ ] x=0 -> 0.8:(x'=0) + 0.2:(x'=1);
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```

- We use it to describe the set of possible states and transitions between them.

The PRISM Modelling Language

- Formulas and Labels:

```
formula num_tokens = q1+q2+q3+q4+q5;  
formula crash = x1=x2 & y1=y2;  
label "crashed" = crash  
//[moveNorth] !crash & ... -> ...;
```

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[ ] move=0 & ... -> ... & (move'=1);  
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etc.
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- Turn-based behaviour:

```
[ ] move=0 & ... -> ... & (move'=1);
[ ] move=1 & ... -> ... & (move'=2);
etc.
```

- Rewards:

```
rewards
x>0 & x<10 : 2*x;
x=10 : 100;
[a] true : x;
[b] true : 2*x;
endrewards
```


The PRISM Modelling Language

- Modelling language allows to design models in a code-like style
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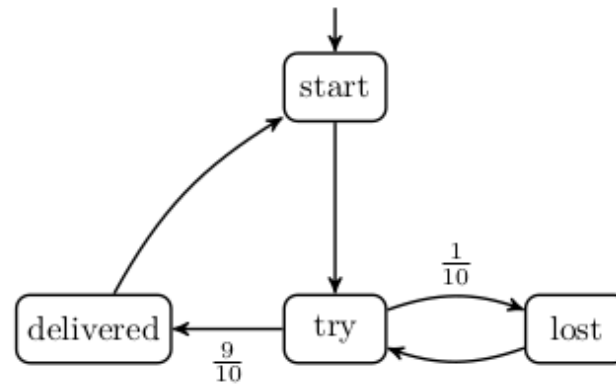
Other concepts include:

- Module Renaming

```
module Proc2 = Proc1 [ idle2=idle1, ... ] endmodule
```

- Synchronization between modules
- Partially Observable Models
- Continuous-time Models
- Process Algebra Operators

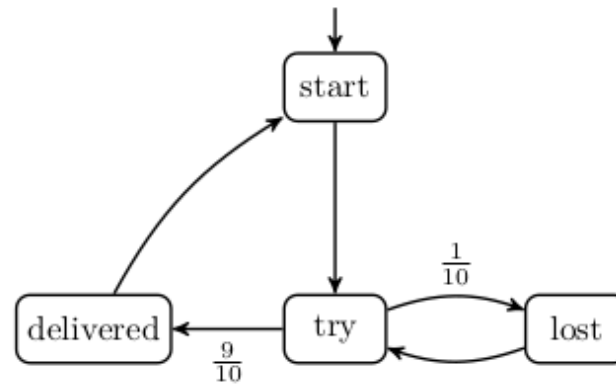
Communication Protocol



dtmc

...

Communication Protocol

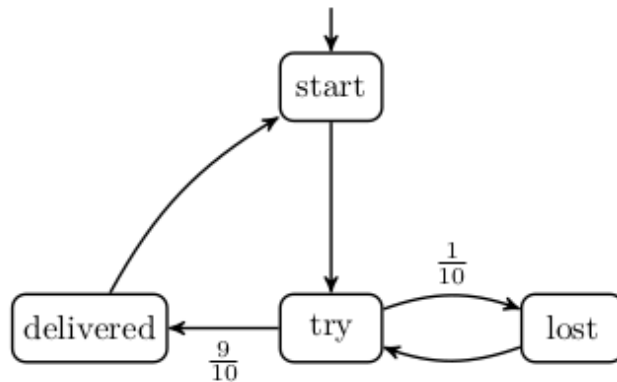


dtmc

...

Live Coding!

Communication Protocol



dtmc

```
label "success" = delivered=1;
label "lost" = lost=1;
```

```
module msg_delivery
```

```
start: [0..1] init 1;
```

```
try: [0..1] init 0;
```

```
lost: [0..1] init 0;
```

```
delivered: [0..1] init 0;
```

```
[ ] start=1 -> 1: (start'=0) & (try'=1);
```

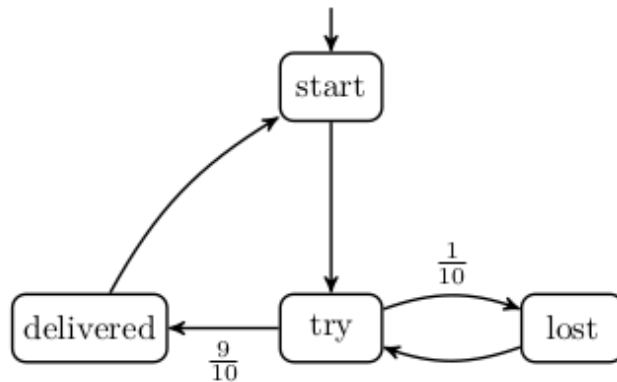
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[ ] try=1 -> 0.1: (try'=0) & (lost'=1) +
0.9: (try'=0) & (delivered'=1);
```

```
[ ] lost=1 -> 1: (lost'=0) & (try'=1);
```

```
[ ] delivered=1 -> 1: (delivered'=0) & (start'=1);
```

```
endmodule
```

Communication Protocol with Counting



```

dtmc

label "success" = delivered=1;
label "lost" = lost=1;

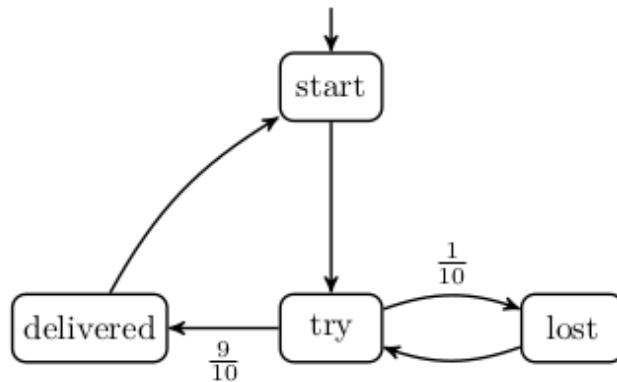
...

module msg_delivery

endmodule
  
```

Live Coding!

Communication Protocol with Counting



dtmc

```
label "success" = delivered=1;
label "lost" = lost=1;
```

```
const int MAX_COUNT;
```

```
module msg_delivery
```

```
start: [0..1] init 1;
try: [0..1] init 0;
lost: [0..1] init 0;
delivered: [0..1] init 0;
delivered_count: [0..MAX_COUNT] init 0;
lost_count: [0..MAX_COUNT] init 0;
```

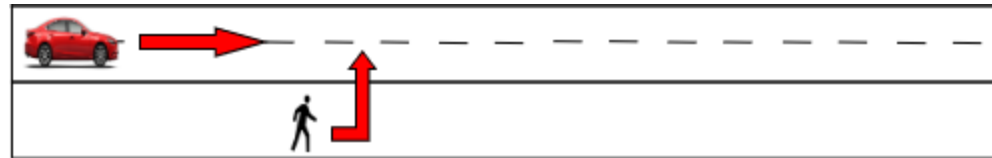
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[ ] try=1        -> 0.1: (try'=0) & (lost'=1) +
                  0.9: (try'=0) & (delivered'=1);
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```
[ ] lost=1       & lost_count<MAX_COUNT      -> 1: (lost'=0) & (try'=1) & (lo
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```

endmodule

Simulating Urban Environments



```
dtmc
...

module car
  // x and y coordinates, velocity

endmodule

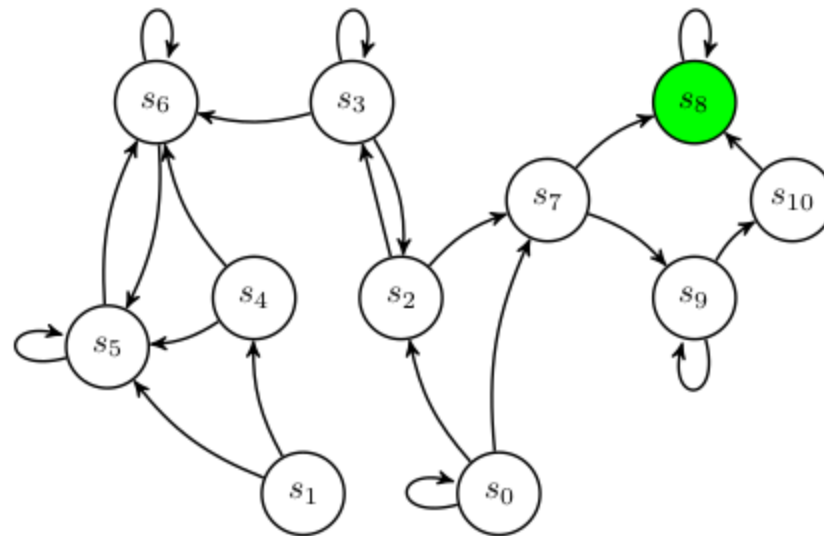
module pedestrian
  // x and y coordinates, viewing direction in {left, right, north}

endmodule
```


Probabilistic Reachability

- We start with objectives similar to the ones discussed at the beginning of the semester:

What is the probability that our system reaches its goal state?



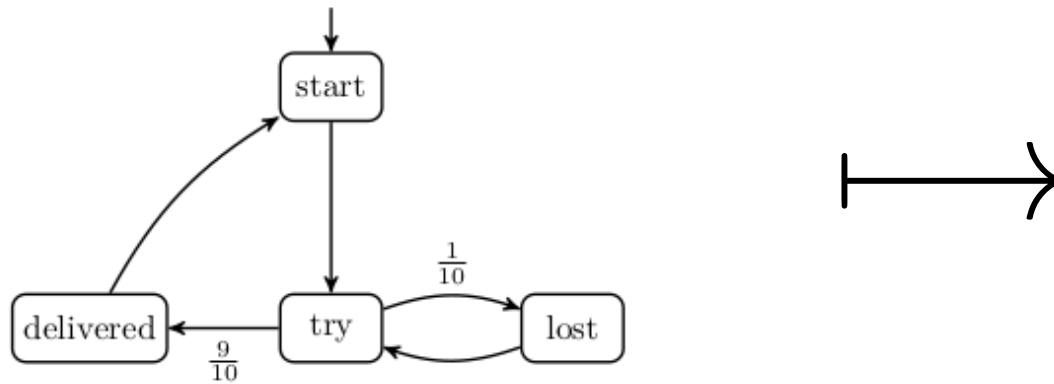
Before we talk about Algorithms...

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How can we represent a MC in code/memory?

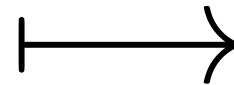
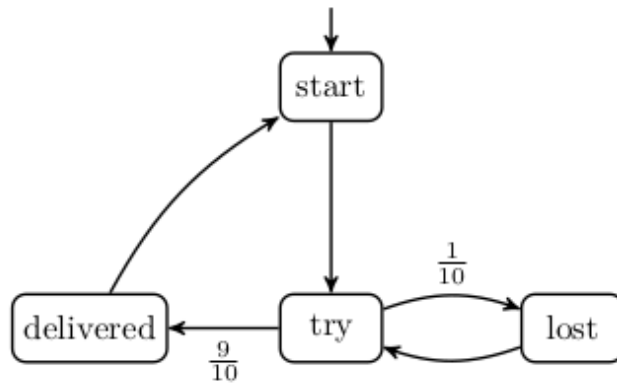
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Before we talk about Algorithms...

How can we represent a MC in code/memory?



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & \frac{9}{10} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Model Checking with Markov Chains

- Explicit CTL model checking allows *qualitative* model checking.
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Model Checking with Markov Chains

- Explicit CTL model checking allows *qualitative* model checking.
 - $\mathcal{M}, start \models \exists \mathbf{G} \neg delivered$?
- We want to do *quantitative* model checking.
 - How *likely* is the system to fail?

$$Pr(\mathcal{M}, s \models \mathbf{F} s_{error})$$

- Whats the *probability* of my message to arrive after infinitely many tries?

$$Pr(\mathcal{M}, s \models \mathbf{F} delivered)$$

Paths

- A path $\pi = s_0 s_1 s_2 \dots \in S^\omega$, s.t. $\mathbb{P}(s_i, s_{i+1}) > 0, \forall i \geq 0$
- $Paths(\mathcal{M})$ is the set of all paths in \mathcal{M} and
- $Paths_{fin}(\mathcal{M})$ is the set of all finite path fragments in \mathcal{M} .

Events and Paths

In order to talk about probabilities of certain paths we need to briefly touch probability spaces.

- Outcomes = $\{HH, HT, TH, TT\}$
- Events = $\{HH\}, \{HT\}, \{TH\}, \{TT\}$

We could, for example, be interested in the events where H is thrown first = $\{HH\}, \{HT\}$.

What is a possible outcome in a specific Markov Chain \mathcal{M} ?

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→ an infinite path $\pi \in Paths(\mathcal{M})!$

- Outcomes = $Paths(\mathcal{M})$
- Events of interest are $\hat{\pi}_1, \hat{\pi}_2, \dots \in Paths_{fin}(\mathcal{M})$ that satisfy our property
- Formally we introduce the *cylinder set* of a prefix:

$$Cyl(\hat{\pi}_i) = \{\pi \in Paths(\mathcal{M}) \mid \hat{\pi}_i \in \text{pref}(\pi)\}$$

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$$Cyl(\hat{\pi}_i) = \{\pi \in Paths(\mathcal{M}) \mid \hat{\pi}_i \in \text{pref}(\pi)\}$$

- The probability of one event of interest is then:

$$Pr(Cyl(\hat{\pi}_i)) = Pr(Cyl(s_0 s_1 \dots s_n)) = \prod_{0 \leq i < n} \mathbb{P}(s_i, s_{i+1})$$

Reachability Probabilities

Let $B \subseteq S$ be a set of states. We are interested in

$$Pr(\mathcal{M}, s_0 \models \mathbf{F}B).$$

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We can characterize all path fragments π that satisfy $\mathbf{F}B$ with the set

$$\Pi_{\mathbf{F}B} = Paths_{fin}(\mathcal{M}) \cap (S \setminus B)^* B$$

All $\hat{\pi} \in \Pi_{\mathbf{F}B}$ are pairwise disjoint, hence:

$$Pr(\mathcal{M}, s_0 \models \mathbf{F}B) = \sum_{\hat{\pi} \in \Pi_{\mathbf{F}B}} Pr(Cyl(\hat{\pi}))$$

Computing $Pr(\mathcal{M}, s_0 \models C \cup B)$

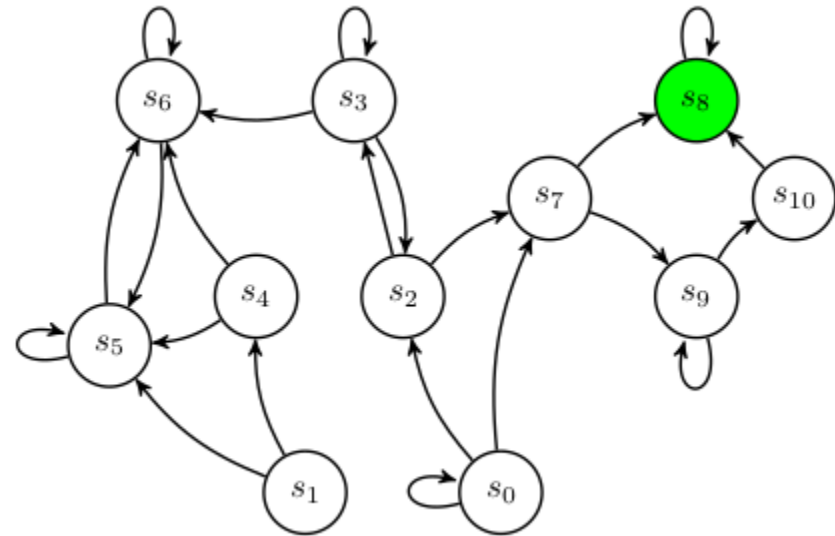
- We know that $\mathbf{F}B \equiv C \cup B$, with $C = S$ or simply '*true* $\cup B$ '
 - Develop algorithm for arbitrary C

Computing $Pr(\mathcal{M}, s_0 \models C \cup B)$

- We know that $\mathbf{FB} \equiv C \cup B$, with $C = S$ or simply ' $true \cup B$ '
 - Develop algorithm for arbitrary C

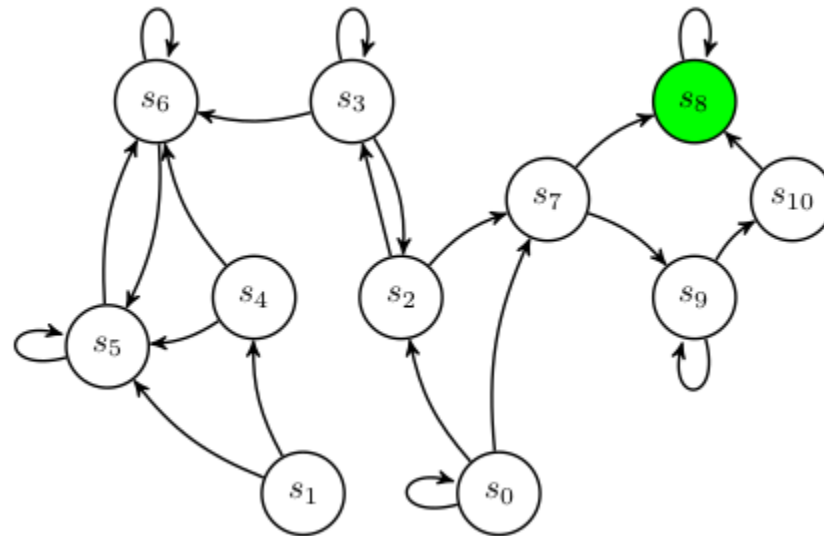
2-step algorithm:

- 1) Identify three disjoint subsets of S :
 - $S_{=1}$: The set of states with probability of 1 to reach B .
 - $S_{=0}$: The set of states with probability of 0 to reach B .
 - $S_?$: The set of states with probability $\in (0, 1)$ to reach B .



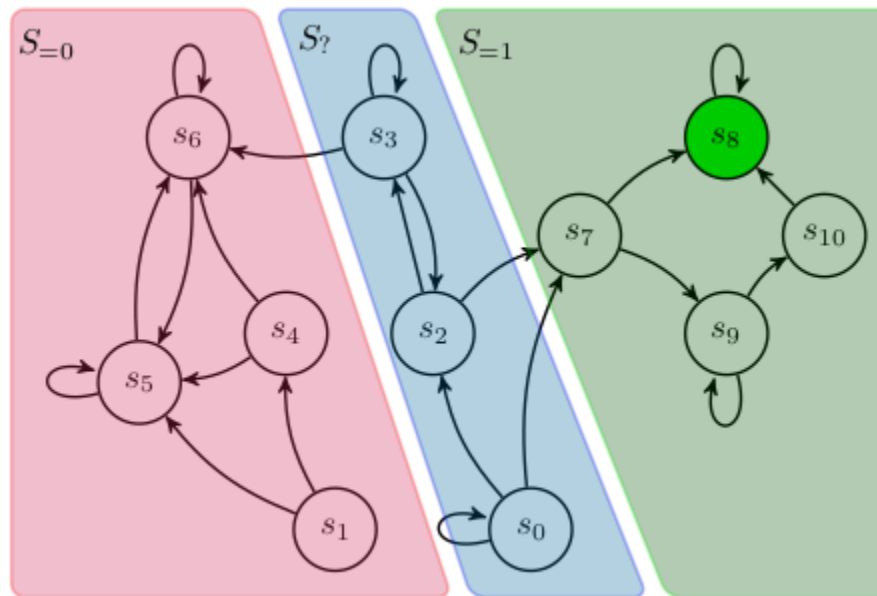
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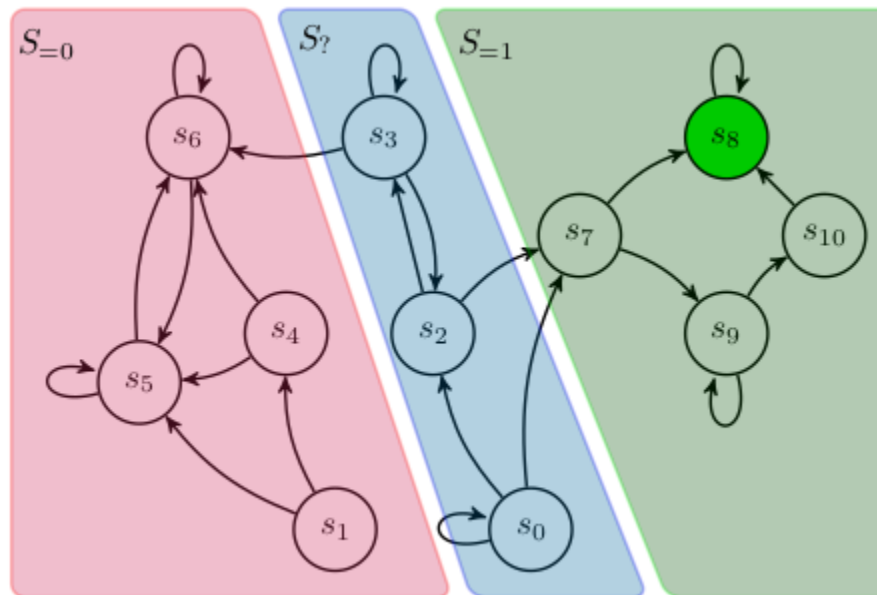
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FAIK

Computing $S_?$

We are left with computing the probabilities for $s \in S_?$



Computing S ?

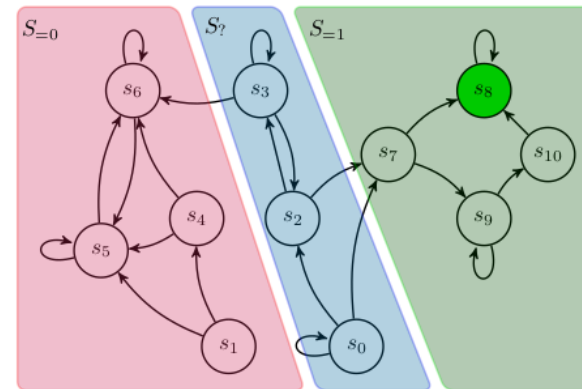
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Computing $S_?$

We are left with computing the probabilities for $s \in S_?$

- The probability to reach $S_{=1}$ in one step: $\sum_{u \in S_{=1}} \mathbb{P}(s, u)$
- and the probability to reach $S_{=1}$ via a path fragment $(s \ t \ \dots \ u)$: $\sum_{t \in S_?} \mathbb{P}(s, t) \cdot x_t$
- Together

$$x_s = \sum_{t \in S_?} \mathbb{P}(s, t) \cdot x_t + \sum_{u \in S_{=1}} \mathbb{P}(s, u)$$



Computing $S_?$

Let us rewrite this into matrix notation:

- $A_? = (\mathbb{P}(s, t))_{s, t \in S_?}$
- $x = (x_s)_{s \in S_?}$
- $b = (\sum_{u \in S_{-1}} \mathbb{P}(s, u))_{s \in S_?}$

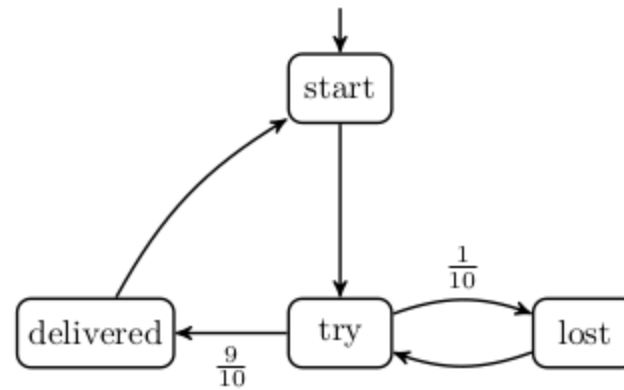
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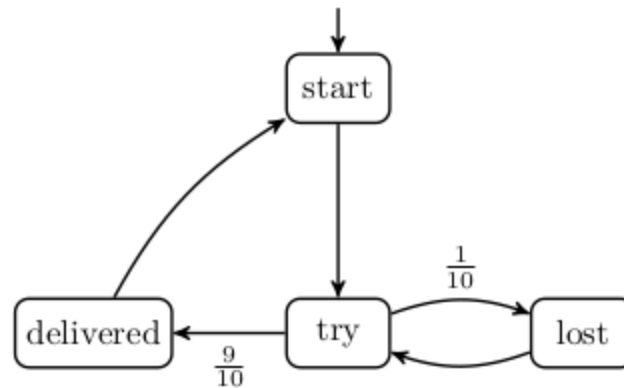
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$$x_s = \sum_{t \in S_?} \mathbb{P}(s, t) \cdot x_t + \sum_{u \in S_{=1}} \mathbb{P}(s, u) \rightsquigarrow x = A_? \cdot x + b = (I - A_?) \cdot x = b$$

Communication Protocol



Communication Protocol



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ \frac{9}{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{10} \\ 0 & -1 & 1 \end{bmatrix} \cdot \mathbf{x} = \begin{pmatrix} 0 \\ \frac{9}{10} \\ 0 \end{pmatrix} \rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Done

Transient State Probabilities

We will consider a slightly different algorithm:

$$\mathbf{A}^n = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdots \mathbf{A}$$

contains the probability to be in state t after n steps in entry $\mathbf{A}^n(s, t)$.

We call

$$\Theta_n^{\mathcal{M}}(t) = \sum_{s \in \mathcal{S}} \mathbf{A}^n(s, t)$$

the *transient state probability* for state t .

Transient State Probabilities

Let's consider $(\Theta_n^{\mathcal{M}}(t))_{s \in S}$, the vector of transient state probabilities for the n th step.

We can compute $Pr(\mathcal{M}, s_0 \models \mathbf{F}^{\leq n} B)$ in a modified Markov chain:

$$\mathcal{M}_B = (S, s_0, \mathbb{P}_B, AP, L)$$

where:

- $\mathbb{P}_B(s, t) = \mathbb{P}(s, t)$ if $s \notin B$
- $\mathbb{P}_B(s, s) = 1$ if $s \in B$
- $\mathbb{P}_B(s, t) = 0$ if $s \in B$ and $t \notin B$

i.e. all $s \in B$ become sinks and B cannot be left anymore.

Transient State Probabilities

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i.e. all $s \in B$ become sinks and B cannot be left anymore.

We then have

$$Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = Pr(\mathcal{M}_B, s \models \mathbf{F}^{\leq n} B)$$

and therefore

$$Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = \sum_{t \in B} \Theta_n^{\mathcal{M}_B}(t)$$

Computing $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B)$ via Transient State Probabilities

We have the following algorithm to compute $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B)$:

- $\Theta_0^{\mathcal{M}}(t) = \mathbf{e}_i$, i.e. the unit vector with 1 at the i th position and 0 else.
- For $k = 0$ up to $n - 1$: $\Theta_{k+1}^{\mathcal{M}}(t) = \mathbf{A} \cdot \Theta_k^{\mathcal{M}}(t)$
- $Pr(\mathcal{M}, s \models \mathbf{F}^{\leq n} B) = \sum_{t \in B} \Theta_n^{\mathcal{M}_B}(t)$