Probabilistic Model Checking

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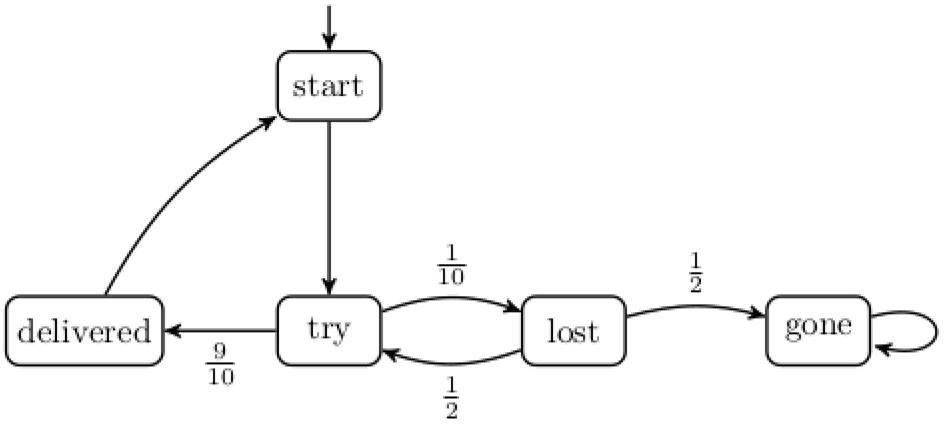
Communication Protocol with Faults



$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{10} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \cdot \mathbf{x} = \begin{pmatrix} 0 \\ \frac{9}{10} \\ 0 \end{pmatrix} \rightarrow \mathbf{x} = \begin{pmatrix} \frac{18}{19} \\ \frac{18}{19} \\ \frac{9}{19} \end{pmatrix}$$

Cowboy Shootout

- The three may shoot as long as anyone else is still alive. Due to differences in (re)loading times, we assume they shoot in turns. That is, The Good shoots first, then The Bad and finally The Ugly.
- The Good has a chance of a half of hitting anyone. If he hits, he does so uniformly over the living contestants.
- The Bad has a chance of 0.9 of hitting anyone. If The Ugly is alive, then he aims for him. If The Ugly already died, then he aims at The Good.
- The Ugly hits either no one or one of the living contestants and he does so with a uniform probability over these events.

```
module shootout
  cowboy: [1..3] init 1;
  good: bool init true;
  bad: bool init true;
  ugly: bool init true;
  [] cowboy=1 & good & bad & ugly   -> 1/2 :(cowboy'=2) +
                                       1/4 :(bad'=false) & (cowboy'=3) +
                                       1/4 :(ugly'=false) & (cowboy'=2);
  [] cowboy=1 & good & bad & !ugly  -> 1/2 :(cowboy'=2) +
                                       1/2 :(bad'=false) & (cowboy'=1);
  [] cowboy=1 & good & !bad & ugly  -> 1/2 :(cowboy'=3) +
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  [] cowboy=2 & good & bad & ugly   -> 0.1 :(cowboy'=3) +
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  [] cowboy=2 & good & bad & !ugly  -> 0.1 :(cowboy'=1) +
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  [] cowboy=3 & good & bad & ugly  -> 1/3 :(cowboy'=1) +
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  [] !good & !bad &  ugly -> true;
endmodule
```

Recap: Constrained Reachability

- Computing $Pr(\mathcal{M}, s_0 \models C\ \mathbf{U}\ B)$
- We have used a linear equation solver to compute the probability of satisfying the constrained reachability problem.

Probabilistic Computation Tree Logic

Probabilistic Computation Tree Logic [PCTL] is the probabilistic extension of CTL.

- Boolean state representation.
- $\forall$ and $\exists$ are replaced by $\mathrm{Pr}_J(\varphi)$, where $J \subseteq [0,1]$
  - The interpretation for each state $s \in S$: $Pr(\mathcal{M}, s \models \varphi) \in J$

PCTL - Syntax

Subdivision into *state* (Φ)- and *path*-formulae (φ):

$$\begin{aligned}\Phi ::= &\ true \\ \mid &\ a \\ \mid &\ \Phi_1 \wedge \Phi_2 \\ \mid &\ \neg\Phi \\ \mid &\ \Pr_J(\varphi)\end{aligned} \qquad \begin{aligned}\varphi ::= &\ \mathbf{X}\Phi \\ \mid &\ \Phi_1 \ \mathbf{U}\ \Phi_2 \\ \mid &\ \Phi_1 \ \mathbf{U}^{\leq n}\ \Phi_2\end{aligned}$$

where $a \in AP$ and $J \subseteq [0,1]$.

PCTL - Satisfaction Relation

For a given state $s \in S$

$$\begin{aligned}
s &\models a && \text{iff } a \in L(s),\\
s &\models \neg\varphi && \text{iff } s \not\models \varphi,\\
s &\models \varphi \wedge \psi && \text{iff } s \models \varphi \text{ and } s \models \psi,\\
s &\models \Pr_J(\varphi) && \text{iff } Pr(s \models \varphi) \in J
\end{aligned}$$

For paths $\pi \in \mathcal{M}$:

$$\begin{aligned}
\pi &\models \mathbf{X}\varphi && \text{iff} \quad \pi[1] \models \varphi\\
\pi &\models \varphi \,\mathbf{U}\, \psi && \text{iff} \quad \exists j \geq 0.\ (\pi[j] \models \psi \wedge (\forall 0 \leq k < j.\ \pi[k] \models \varphi)\\
\pi &\models \varphi \,\mathbf{U}^{\leq n} \psi && \text{iff} \quad \exists 0 \leq j \leq n.\ (\pi[j] \models \psi \wedge (\forall 0 \leq k < j.\ \pi[k] \models \varphi)
\end{aligned}$$

Model Checking a PCTL Formula

- Checking the propositional part of PCTL is easy
- How to compute $Pr(\mathcal{M}, s_0 \models C\ \mathbf{U}\ B)$?
  - We solve a linear equation system. ✓
- How to compute $Pr(\mathcal{M}, s_0 \models \mathbf{X}a)$?

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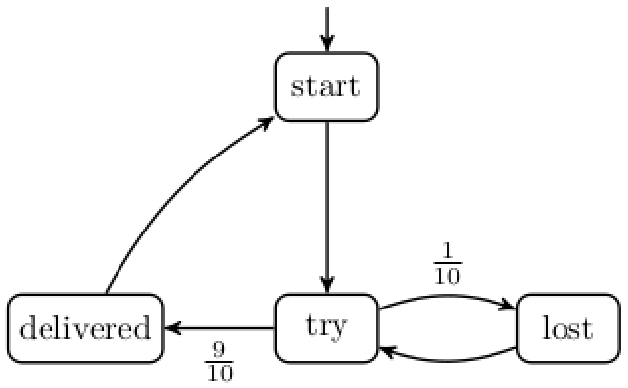
- How can we compute bounded reachabililty: $Pr(\mathcal{M}, s_0 \models \mathbf{F}^{<=k}a)$?
  - Again: Simple Matrix-Vector-Multiplication(s)! ✓

Communication Protocol



- *A message is eventually delivered or lost and our abstraction does not allow faulty values:*

```
"P>=1.0 [ F (delivered=1 | lost=1)] & P>=1.0 [G try<2 ]"
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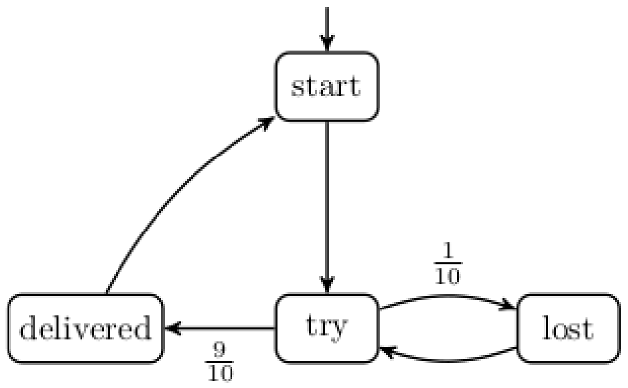
Communication Protocol



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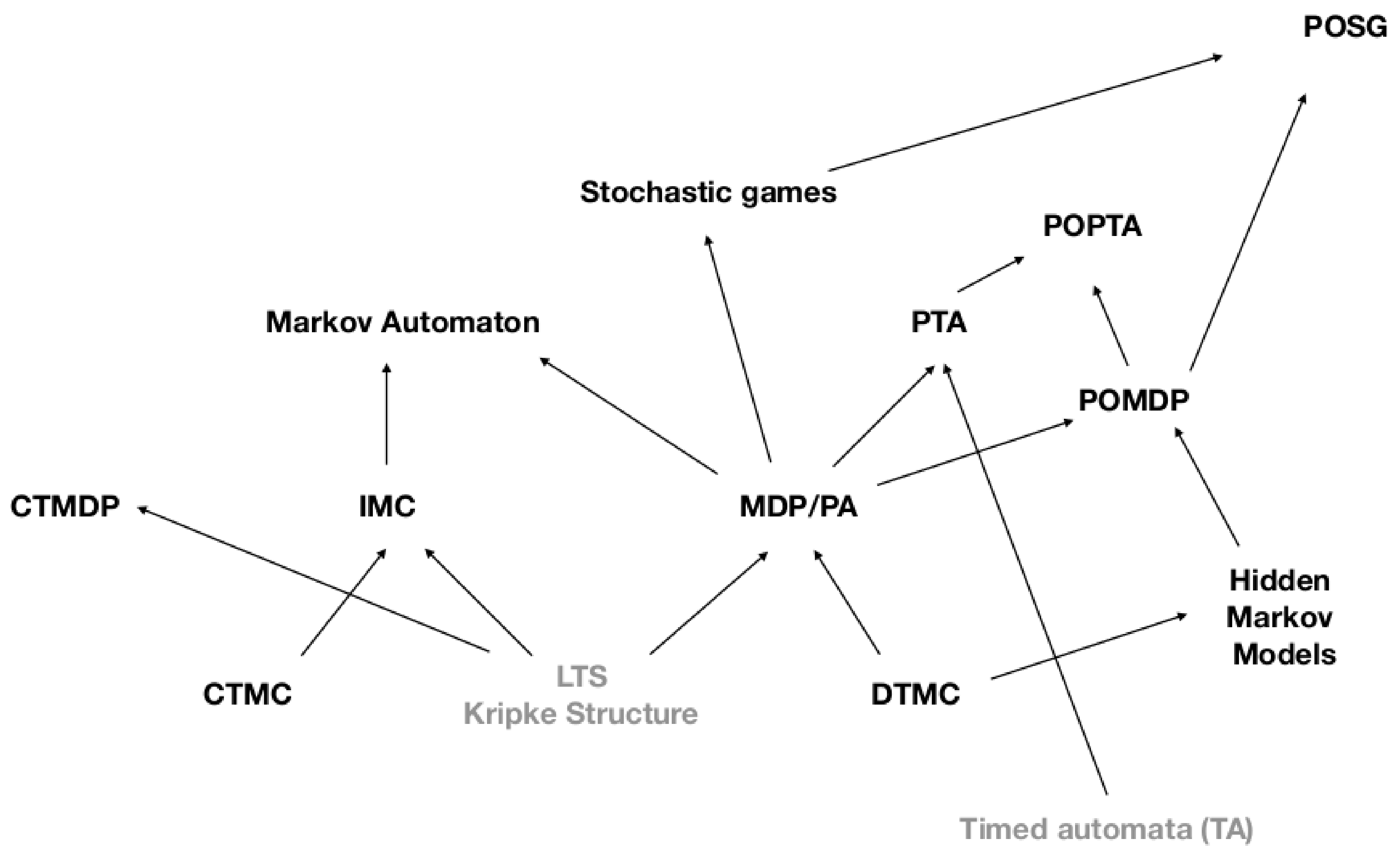
- *A message will almost surely be delivered eventually and trying to send a message implies that with a probability greater or equal 0.99 the message will be sent within three time steps.*

```
"P>=1.0 [ F (delivered=1)] & P>=1.0 [ G(!try=1| P>=0.99 [ (F<=3 delivered=1) ] ) ] "
```

Probabilistic Model Zoo



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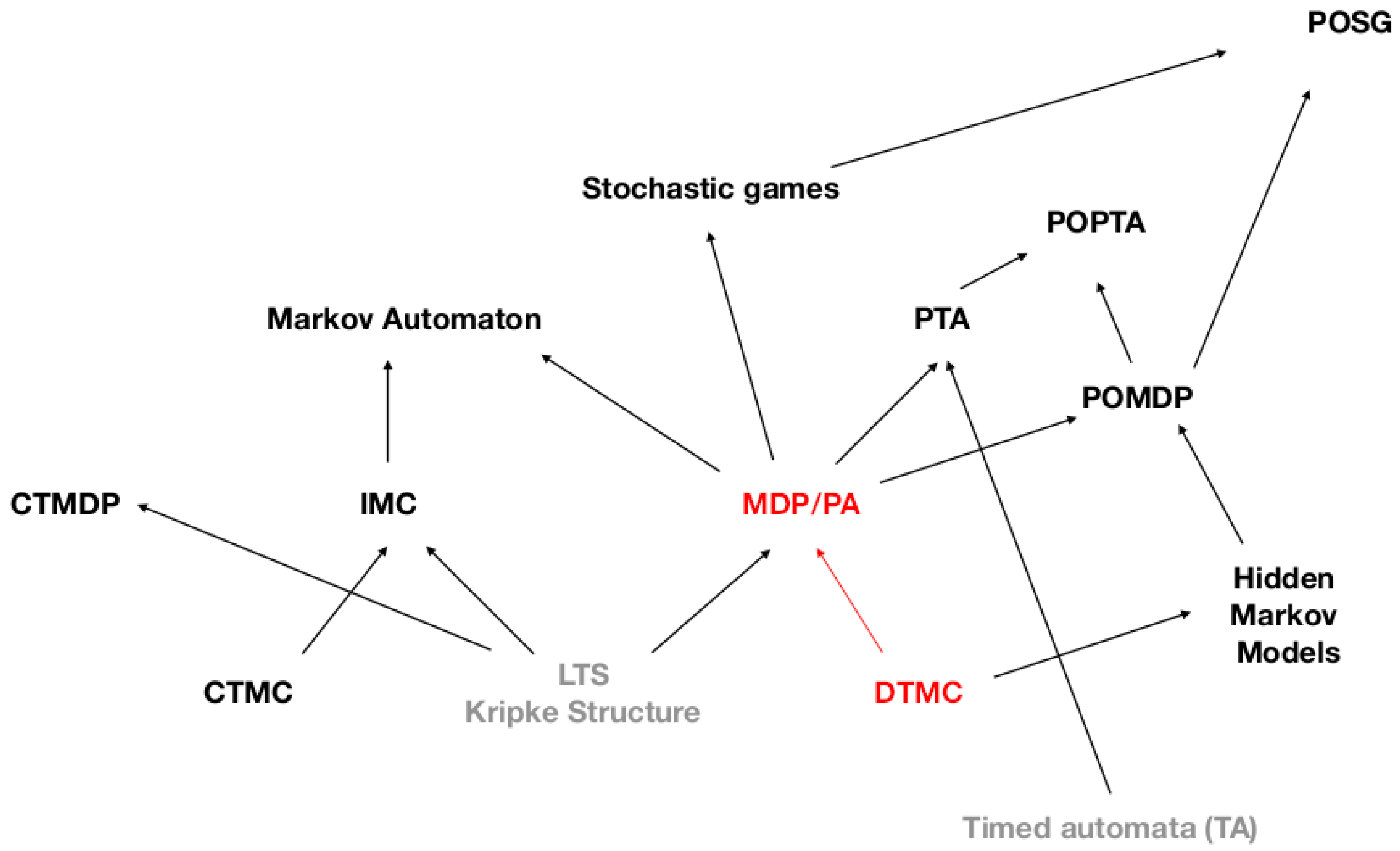

Markov Decision Processes

Markov Decision Process $\mathcal{M} = (S, Act, \mathbb{P}, s_0, AP, L)$

- S a set of states and initial state s_0,
- Act a set of actions,
- $\mathbb{P} : S \times Act \times S \to [0,1]$, s.t.

$$\sum_{s' \in S} \mathbb{P}(s, a, s') = 1 \; \forall (s, a) \in S \times Act$$

- AP set of atomic states and $L : S \to 2^{AP}$ a labelling function.

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The decision a defines the distribution over the next state.

Markov Decision Processes in Code and Memory

- Commands:

```
[moveNorth] x<height -> 0.9: (x'=x+1) + 0.1: true;
[moveEast]  y<width  -> 0.9: (y'=y-1) + 0.1: true;
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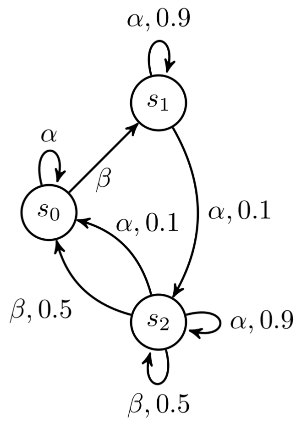

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & \frac{9}{10} & \frac{1}{10} \\ \hline \frac{9}{10} & 0 & \frac{1}{10} \\ \frac{5}{10} & 0 & \frac{5}{10} \end{array}\right]$$

Paths in an MDP

- We extend our definition of a path for an MDP $\mathcal{M}$ as such:
- $\pi = s_0 a_0 s_1 a_1 s_2 a_2 \ldots \in (S \times Act)^\omega$, s.t. $\mathbb{P}(s_i, a_i, s_{i+1}) > 0, \forall i \geq 0$

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- Reasoning about events in an MDP resorts to the resolution of any non-determinism
  - This is done by the use of schedulers (also called strategies/policies/adversaries).

Schedulers

- A scheduler is a function that given the history of the current path returns a distribution over actions to be taken:

$$\sigma : S^* \times S \to Distr(Act)$$

- For simple properties such as reachability so called memoryless deterministic scheduler suffice:

$$\sigma : S \to Act$$

  - This means that the scheduler σ fixes an actions for each state.
  - We can then define the probability of prop under sched

$$Pr^{\sigma}(\mathcal{M}, s \models \mathbf{F} B)$$

Induced Markov Chain

Consider an MDP $\mathcal{M}$ and a memoryless deterministic scheduler:

$$\sigma : S \to Act$$

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Consider an MDP $\mathcal{M}$ and a memoryless deterministic scheduler:

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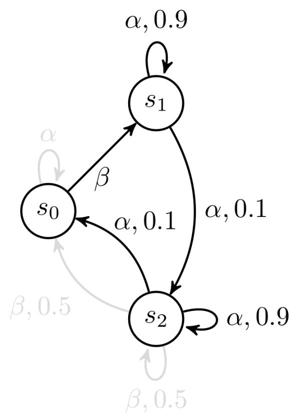
$$\begin{aligned} s_0 &\mapsto \beta \\ s_1 &\mapsto \alpha \\ s_1 &\mapsto \alpha \end{aligned}$$



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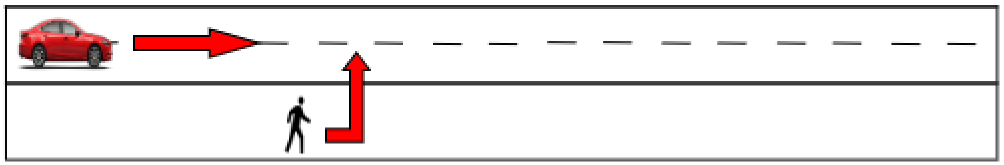
Coding Example



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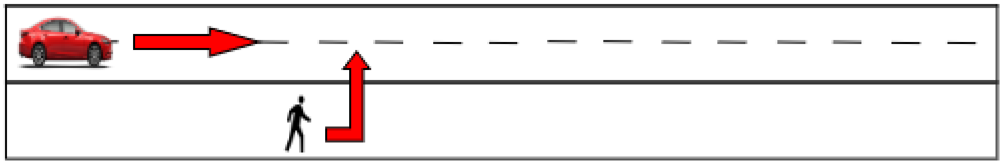

- We introduce velocity and let the car decide whether to
    - switch lanes,
    - accelerate or
    - decelerate.

Reachability in MDPs

- We have introduced nondeterminism into probabilistic models
- Schedulers might maximize oder minimize the probability to satisfy a given property

Reachability in MDPs

- We have introduced nondeterminism into probabilistic models
- Schedulers might maximize oder minimize the probability to satisfy a given property
- We describe this with
  - $Pr^{max}(\mathcal{M}, s \models \mathbf{F}B) = sup_\sigma Pr^\sigma(\mathcal{M}, s \models \mathbf{F}B)$
  - $Pr^{min}(\mathcal{M}, s \models \mathbf{F}B) = inf_\sigma Pr^\sigma(\mathcal{M}, s \models \mathbf{F}B)$

Computing Maximum Reachability Probabilities in MDPs

We want to compute $(x_s) = Pr^{max}(\mathcal{M}, s \models \mathbf{F}B)$ using the following equation system:

- If $s \in B$: $x_s = 1$
- If $s \not\models \exists\mathbf{F}B$: $x_s = 0$
- If $s \notin B$ and $s \models \exists\mathbf{F}B$
    - $x_s = \max\{\sum_{s' \in S} \mathbb{P}(s, a, s') \cdot x_{s'} | a \in Act(s)\}$

- Such that $\sum_{x \in S} x_s$ is minimal.

Value Iteration - Method I

- Approximative method:
  - Compute the probability to reach B after n steps
  - Start with $n = 0$ and stop after some termination criterion is met

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More specifically:

$$\begin{aligned}
x_s^{(0)} &= 1, \forall s \in B \\
x_s^{(n)} &= 0, \forall s \in S_{=0} \\
x_s^{(0)} &= 0, & \forall s \in S \setminus S_{=0} \\
x_s^{(n+1)} &= \max\{\textstyle\sum_{s' \in S} \mathbb{P}(s,a,s') \cdot x_{s'} | a \in Act(s)\}, & \forall s \in S \setminus S_{=0}
\end{aligned}$$

Linear Program - Method II

We can also express the problem as a linear program:

- Minimize $\sum_{x \in S} x_s$, such that:
  - $0 \leq x_s \leq 1$,
  - $x_s = 1$, if $s \in B$,
  - $x_s = 0$, if $s \not\models \exists \mathbf{F} B$,
  - $x_s \geq \sum_{s' \in S} \mathbb{P}(s, a, s') \cdot x_{s'}$, for all actions $a \in Act(s)$, if $s \notin B$ and $s \models \exists \mathbf{F} B$

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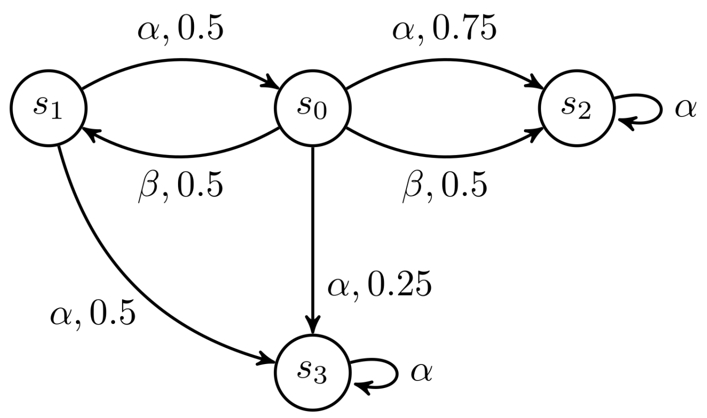

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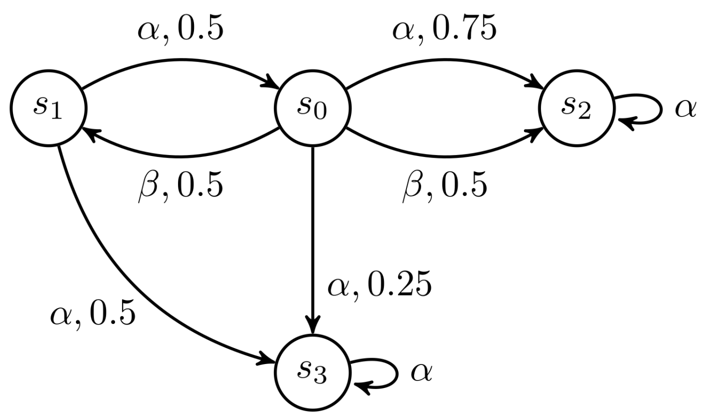

```
var x0 >= 0;
var x1 >= 0;
var x2 >= 0;
var x3 >= 0;

minimize z:     x0+x1+x2+x3;
subject to c0: x0 >= 3/4*x2 + 1/4*x3;
subject to c1: x0 >= 1/2*x1 + 1/2*x2;
subject to c2: x2 = 1;
subject to c3: x3 = 0;
subject to c4: x1 >= 1/2*x0 + 1/2*x3;

subject to c20: x0 <= 1;
subject to c21: x1 <= 1;
subject to c22: x2 <= 1;
subject to c23: x3 <= 1;

end;
```

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- The satisfaction relation for the probabilistic operator needs to be adapted:

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- Syntax for PCTL does not need to be changed
- The satisfaction relation for the probabilistic operator needs to be adapted:
  - We need to consider **all** schedulers:
  - $\mathcal{M}, s \models \mathrm{Pr}_{\leq p}(\varphi)$ iff $Pr^{max}(\mathcal{M}, s \models \varphi) \leq p$
  - $\mathcal{M}, s \models \mathrm{Pr}_{\geq p}(\varphi)$ iff $Pr^{min}(\mathcal{M}, s \models \varphi) \geq p$

Rewards in PRISM

We have already seen rewards in PRISM:

- Rewards:

```
rewards
x>0 & x<10 : 2*x;
x=10 : 100;
[a] true : x;
[b] true : 2*x;
endrewards
```

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- PCTL can be extended to PRCTL:
  - `R=? [ LRA ]`, *What is the long-run average reward?*
  - `R=? [ F "elected" ]`, *What is the accumulated reward until the leader has been elected?*
  - `Rmax=? [ C<=k ]`, *What is the maximum reward that can be achieved within k steps?*
  - ...