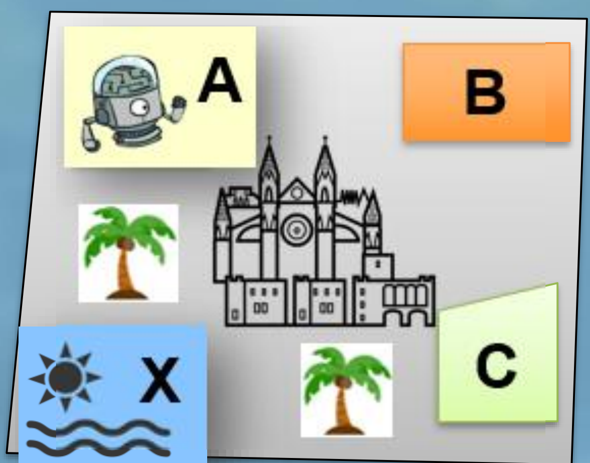
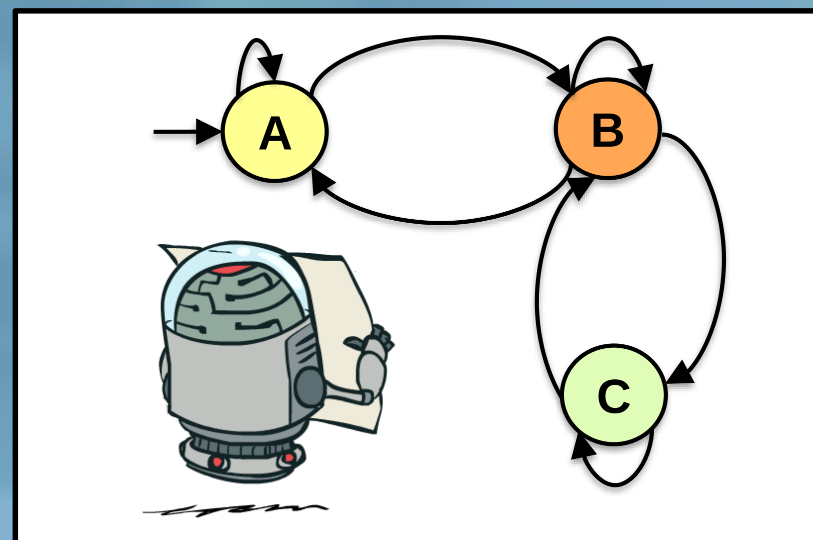
Graz University of Technology
Institute for Applied Information Processing and Communications

Automata and LTL Model Checking

Bettina Könighofer





Model Checking SS21

May 20^{th} 2021

Outline

- Finite automata on finite words
- Automata on infinite words (Büchi automata)
- Deterministic vs non-deterministic Büchi automata
- Intersection of Büchi automata
- Checking emptiness of Büchi automata
- Generalized Büchi automata
- Model checking using automata
- Translation of LTL to Büchi automata
- On-the-fly model checking of LTL

Outline

- Finite automata on finite words
- Automata on infinite words (Büchi automata)
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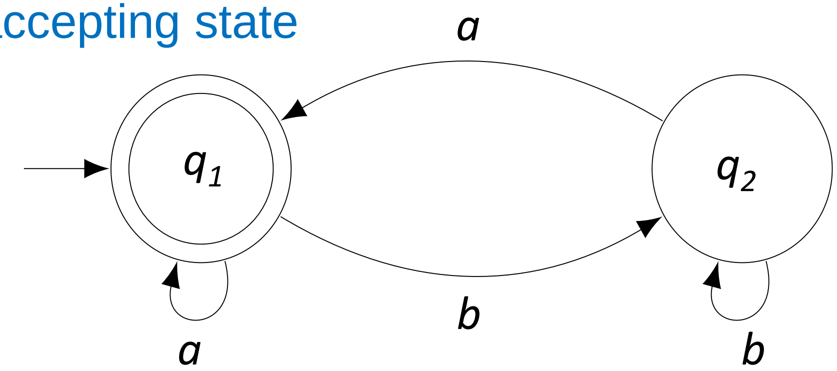
- Model checking using automata

Finite Automata on Finite Words

Regular Automata

- $\mathcal{A} = (\Sigma, Q, \Delta, Q^0, F)$
- Σ is the finite alphabet
- Q is the finite set of states
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- Q^0 is the set of initial states
- F is the set of accepting states
  - $\mathcal{A}$ accepts a word if there is a corresponding run ending in an accepting state



Finite Automata on Finite Words

Regular Automata

- Example: $\mathcal{A} = (\mathbf{\Sigma}, \mathbf{Q}, \mathbf{\Delta}, \mathbf{Q}^0, \mathbf{F})$
- $\mathbf{\Sigma} = \{a, b\}$
- $\mathbf{Q} = \{q_1, q_2\}$
- $\mathbf{\Delta} = \{ (q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_2, b, q_2)\},$
- $\mathbf{Q}^0 = \{q_1\}$
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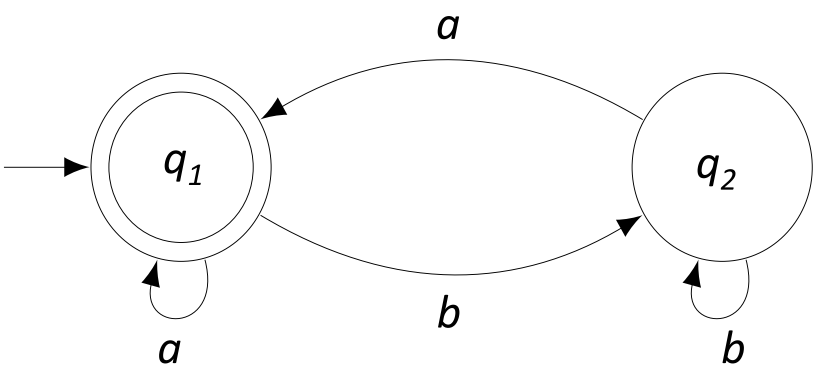

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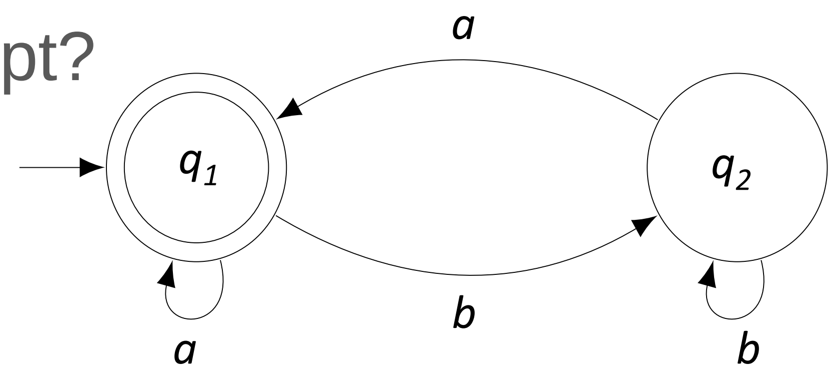
ToDo What words does it accept?



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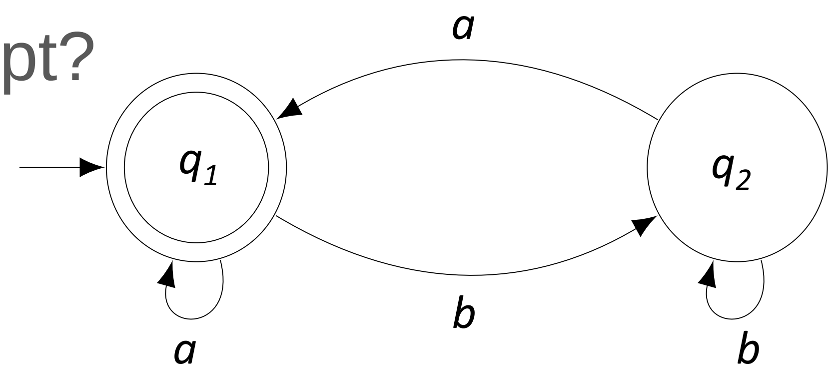

$\mathcal{L}(\mathcal{A})$ = {the empty word} ∪ {all words that end with a}

= $\{\varepsilon\} \cup \{a,b\}^*a$

Finite Automata on Finite Words

Build an automaton that accepts all and only those strings that contain 001

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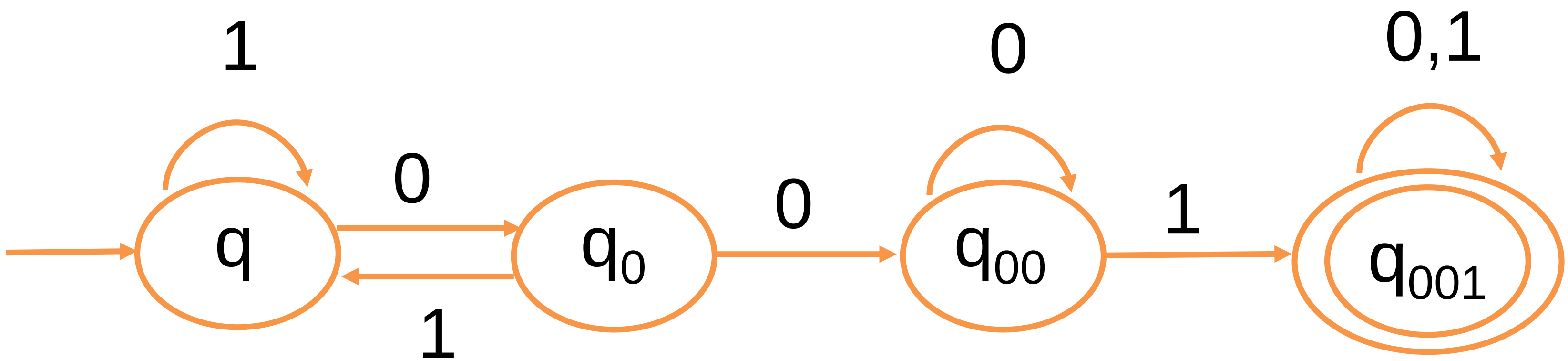

Languages on Finite Automata

- Given a word $v=a_1,a_2,\ldots,a_n$ and automaton $\mathcal{A}$
- A run $\boldsymbol{\rho} = q_0,q_1,\ldots q_n$ of A over v is a sequence of states s.t.:

ToDo

- A run is accepting ⇔
- $\mathcal{A}$ accepts **v** ⇔
- Language of $\mathcal{A}$
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- Languages accepted by finite automata are ***regular languages***.

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  - $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^*$, is the set of words that $\mathcal{A}$ accepts.

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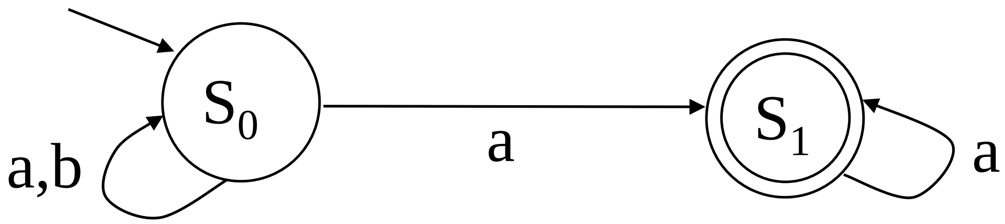
  - $\mathcal{L}(\mathcal{A}) \subseteq \boldsymbol{\Sigma}^*$, is the set of words that $\mathcal{A}$ accepts.
- Languages accepted by finite automata are ***regular languages***.

Deterministic & Non-Deterministic Automata

- $\mathcal{A}$ is deterministic if $\mathbf{\Delta}$ is a function (one output for each input).
  - $|\mathbf{Q}^0| = 1$, and
  - $\forall q \in \mathbf{Q}\ \forall a \in \mathbf{\Sigma}$: $|\ \mathbf{\Delta}(q,a)\ | \leq 1$
- Det. automata have exactly one run for each word.
- Non-det. automata
  - Can have ε-transitions (transitions without a letter)
  - Can have transitions $(q,a,q'),(q,a,q'') \in \mathbf{\Delta}$ and $q'' \neq q'$

Nondeterministic Finite Automata (NFA)

- NFA accepts all words that have a run to an accepting state
- What is the language of this automaton?



Nondeterministic Finite Automata (NFA)

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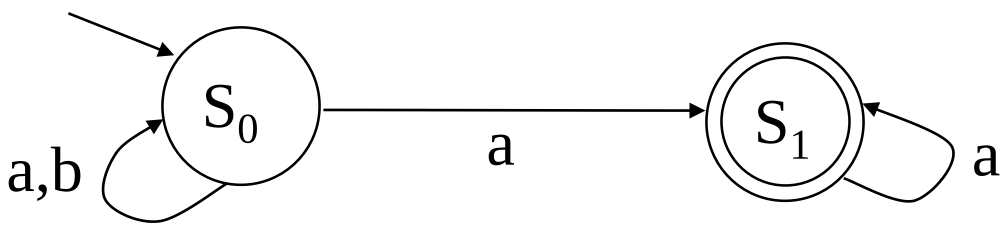
$\mathcal{L}(\mathcal{A})$ = {all words that end with a}

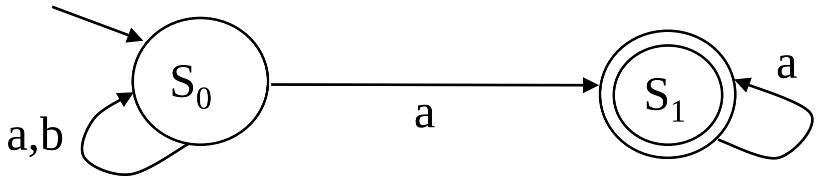


Equivalent deterministic automaton

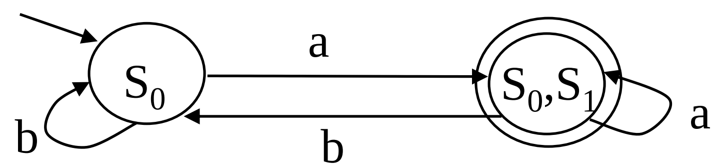
- Every NFA can be transformed to DFA.
  How? Idea only

- Hint:

Non-deterministic automaton $\mathcal{A}$



Equivalent Det. automaton $\mathcal{A}$'



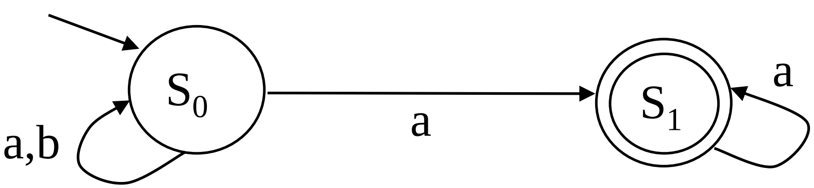
Equivalent deterministic automaton

- Every NFA can be transformed to DFA.
- Subset-Construction (exponential blow-up)
  - NFA: $\mathcal{A} = (\mathbf{\Sigma},\mathbf{Q},\mathbf{\Delta},\mathbf{Q}^0,\mathbf{F})$
  - DFA: $\mathcal{A}' = (\mathbf{\Sigma}, P(\mathbf{Q}),\mathbf{\Delta}',\{\mathbf{Q}^0\},\mathbf{F}')$ such that
    - $\mathbf{\Delta}': P(Q) \times \Sigma \rightarrow P(Q)$ where $(Q_1, a\,, Q_2) \in \Delta'$ if

$$Q_2 = \bigcup_{q \in Q_1} \{q' | (q, a, q') \in \Delta\}$$

    - $F' = \{Q' | Q' \cap F \neq \emptyset\}$

Non-deterministic automaton $\mathcal{A}$



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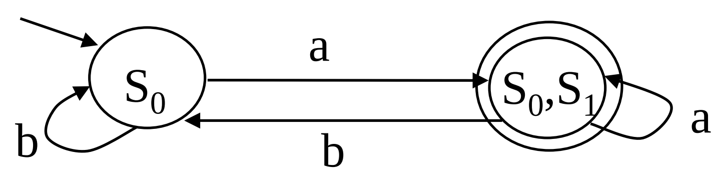

Equivalent deterministic automaton

- Compute the equivalent DFA
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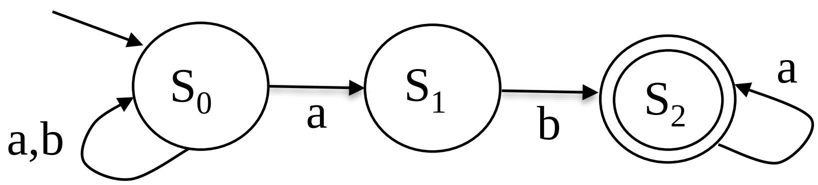

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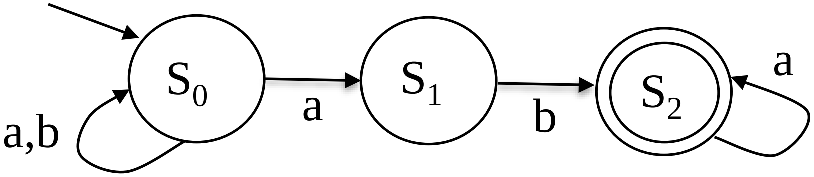

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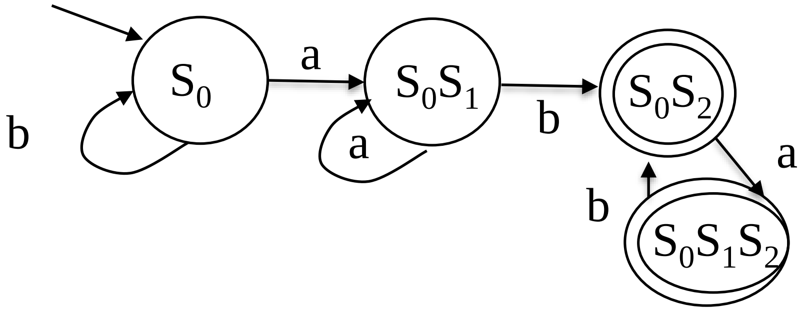

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Non-deterministic automaton $\mathcal{A}$

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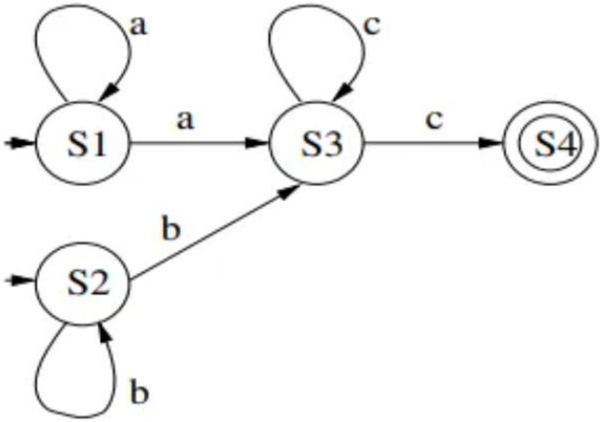

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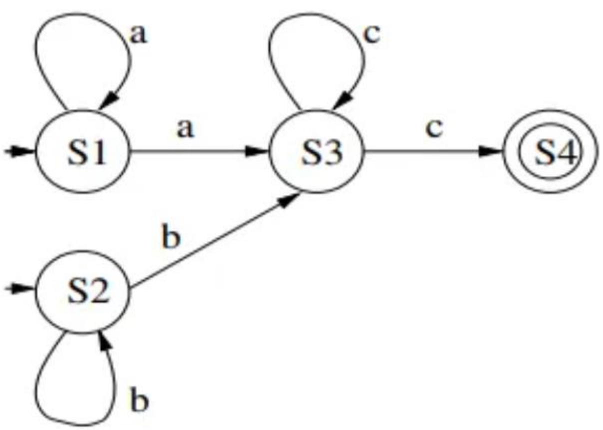

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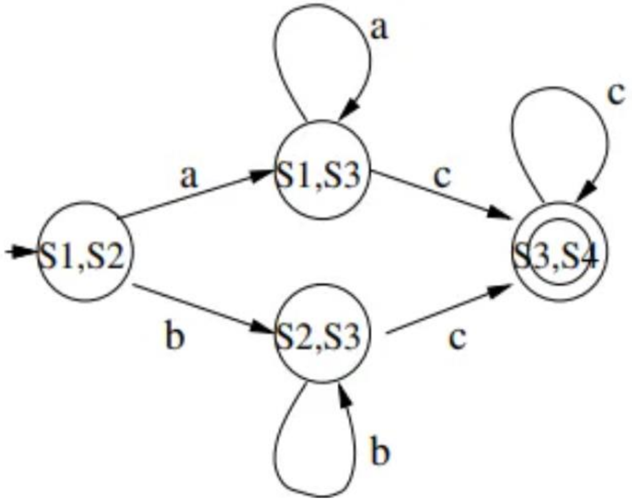


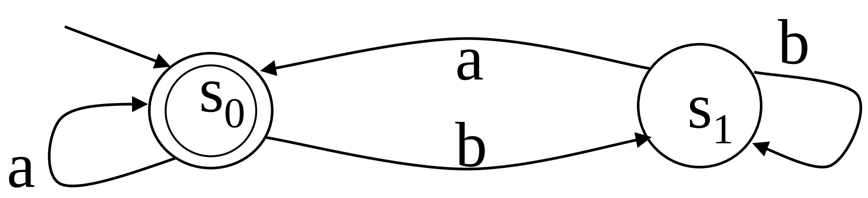
SCOS
Secure & Correct Systems

Complement of DFA

- The complement automaton $\overline{\mathcal{A}}$ accepts exactly those words that are rejected by $\mathcal{A}$

How do we construct $\overline{\mathcal{A}}$?

$\mathcal{A}$

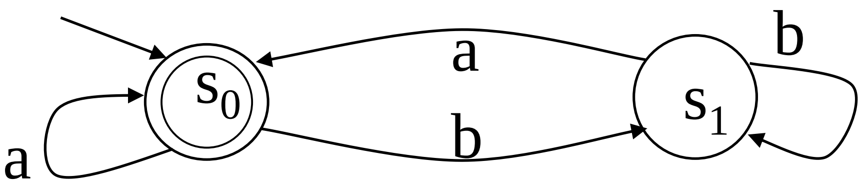


$\overline{\mathcal{A}}$=?

Complement of DFA

- The complement automaton $\bar{\mathcal{A}}$ accepts exactly those words that are rejected by $\mathcal{A}$
- Construction of $\bar{\mathcal{A}}$
  1. Substitution of accepting and non-accepting states

$\mathcal{A}$



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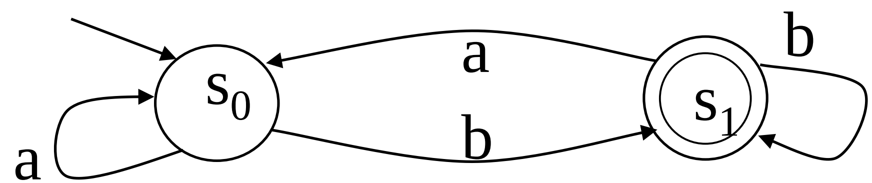
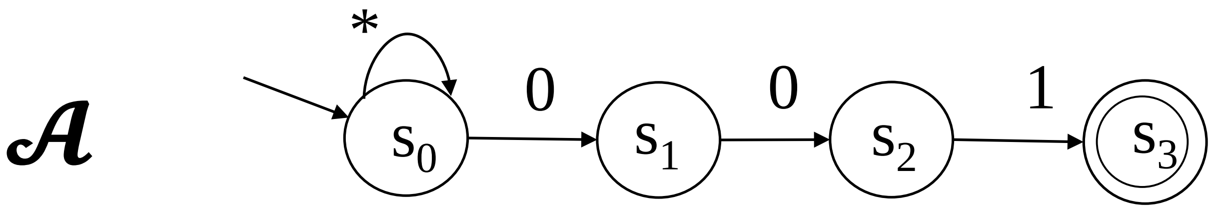


Consider NFA that accepts words that end with 001



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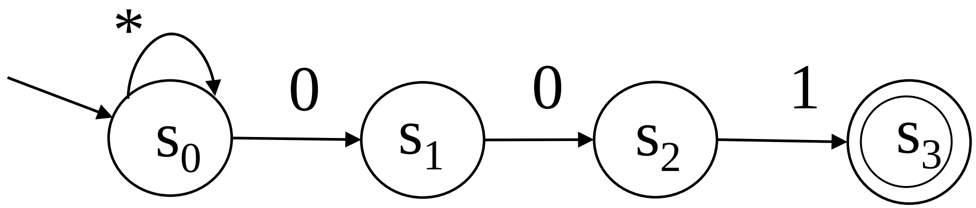
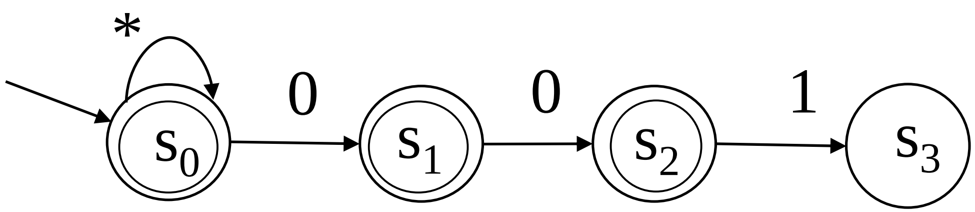


Let’s try switching accepting and non-accepting states:

$\bar{\mathcal{A}}$



Is $\bar{\mathcal{A}}$ the complement of $\mathcal{A}$?

Consider NFA that accepts words that end with 001

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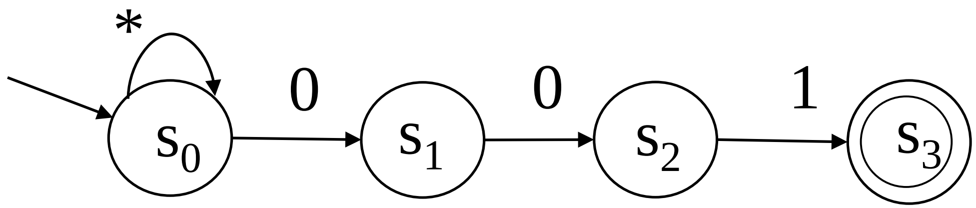

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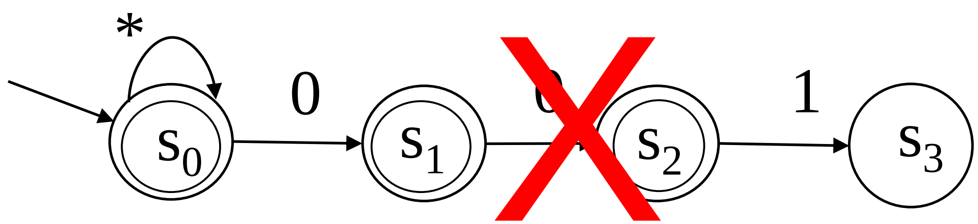

The language of this automaton is {0,1}* - this is wrong!

Complement of NFA

- The complement automaton $\overline{\mathcal{A}}$ accepts exactly those words that are rejected by $\mathcal{A}$
- Construction of $\overline{\mathcal{A}}$
  1. Determinization: Convert NFA to DFA
  2. Substitution of accepting and non-accepting states

Intersections of NFAs

- Given two languages, L_1 and L_2, the intersection of L_1 and L_2 is
  $$L_1 \cap L_2 = \{ w \mid w \in L_1 \text{ and } w \in L_2 \}$$
- Regular languages are closed under Union, Intersection, Concatenation, and Complementation

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- Product automaton of $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ has $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$
  - $Q = Q_1 \times Q_2$ (Cartesian product),
  - $\Delta\big((q_1, q_2), a\big) = (\Delta_1(q_1, a), \Delta_2(q_2, a))$
  - $q_0 = (q_{01}, q_{02})$
  - $(q_1, q_2) \in F$ *iff* $q_1 \in F_1$ *and* $q_2 \in F_2$

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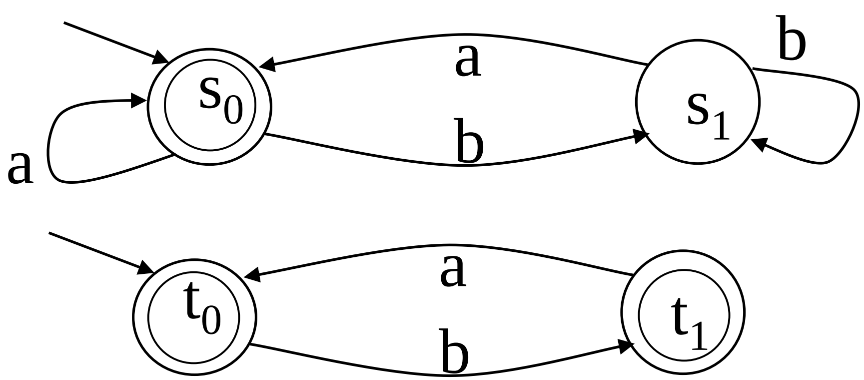
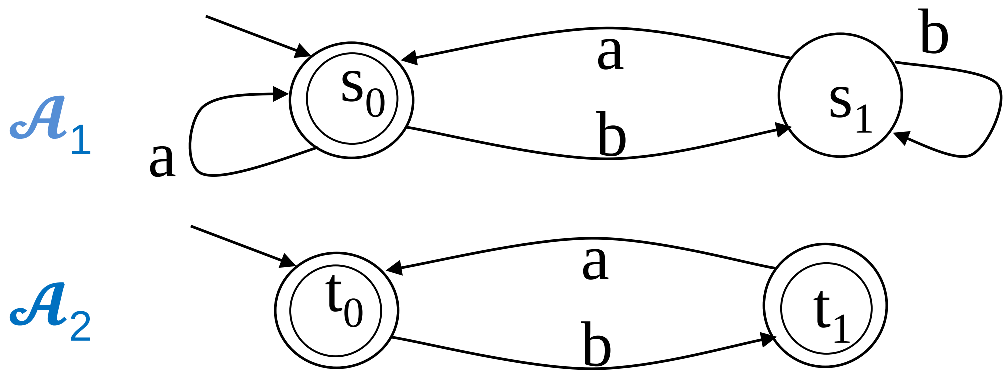
$\mathcal{A}_1$

$\mathcal{A}_2$



$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$

Intersections of NFAs



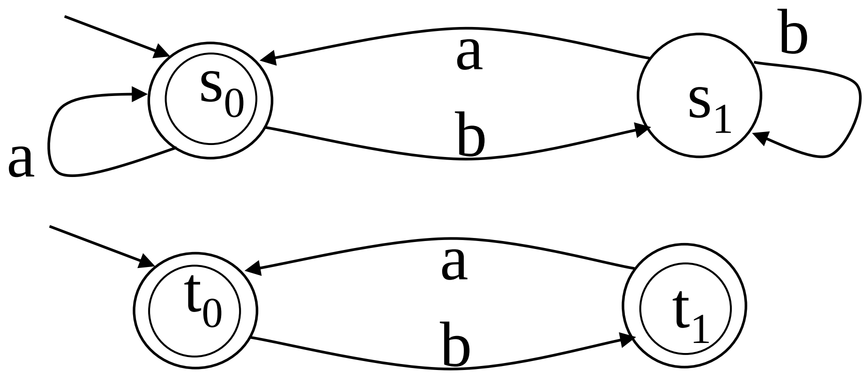
$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$

1. States: (s_0,t_0), (s_0,t_1), (s_1,t_0), (s_1,t_1).
2. Initial state: (s_0,t_0).
3. Accepting states: (s_0,t_0), (s_0,t_1).

Intersections of NFAs

$\mathcal{A}_1$

$\mathcal{A}_2$



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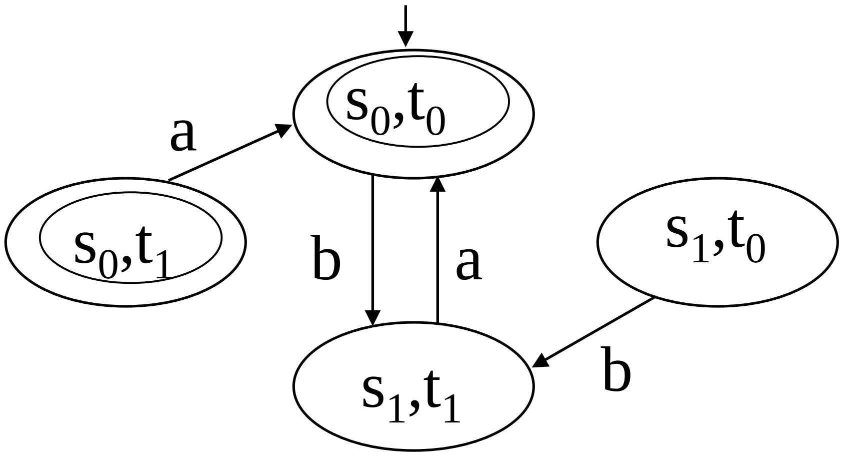
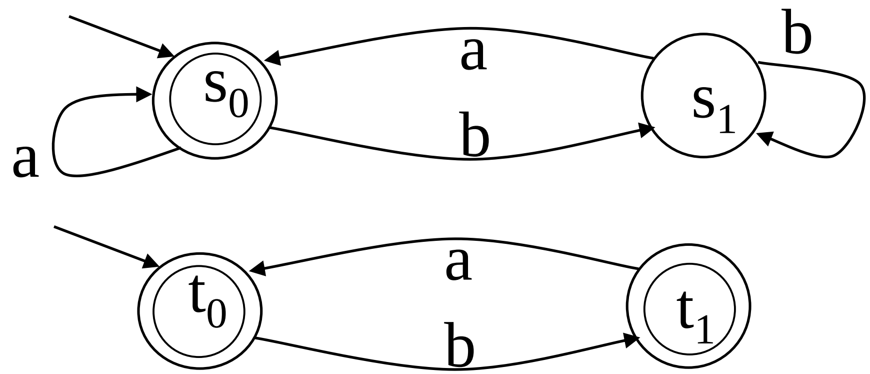


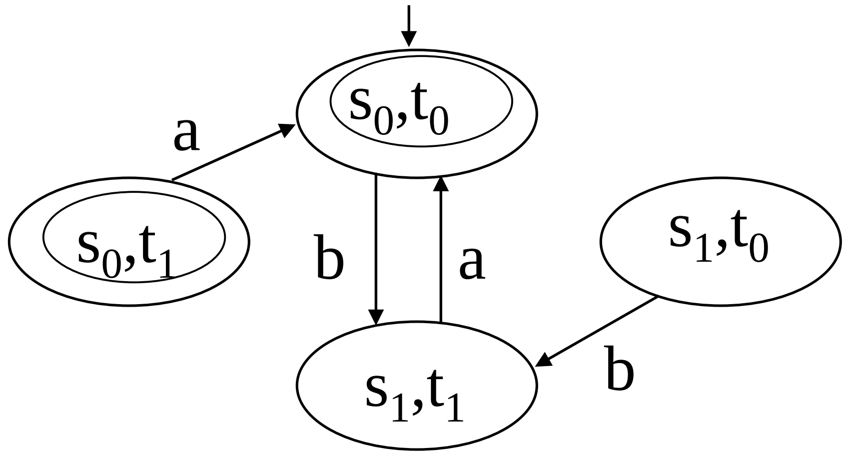
Intersections of NFAs

$L(\mathcal{A}_1) = ?$

$L(\mathcal{A}_2) = ?$

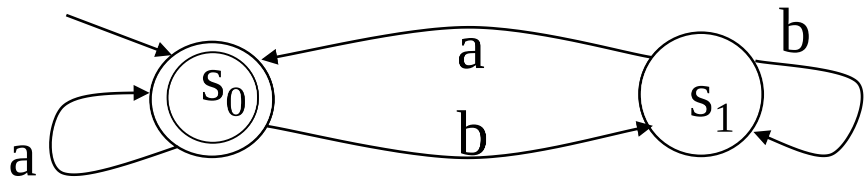


$L(\mathcal{A}_1 \times \mathcal{A}_2) = ?$

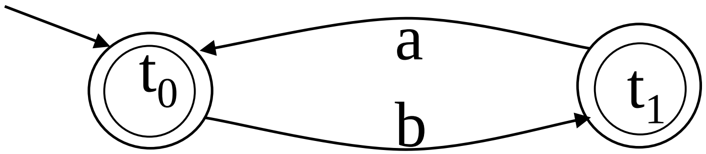


Intersections of NFAs

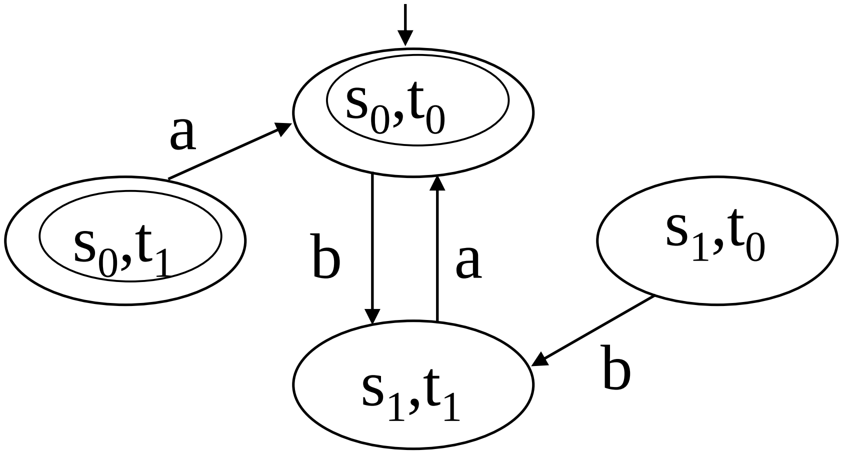
$L(\mathcal{A}_1) = (a+b)^*a + \varepsilon$
(words ending with 'a'
+ empty word)



$L(\mathcal{A}_2) = (ba)^* + (ba)^*b$



$L(\mathcal{A}_1 \times \mathcal{A}_2) = (ba)^*$



Outline

- Finite automata on finite words
- Automata on infinite words (Büchi automata)
- Deterministic vs non-deterministic Büchi automata
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Automata on Infinite Words (Büchi)

$$\mathcal{B} = (\Sigma, Q, \Delta, Q^0, F)$$

- An infinite run ρ is accepting $\Leftrightarrow$ it visits an accepting state an infinite number of times.
  - $\mathrm{inf}(\rho) \cap F \neq \emptyset$

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- $\mathcal{L}(\mathcal{B}) \subseteq \Sigma^\omega$ is the set of all infinite words that $\mathcal{B}$ accepts
- Languages accepted by finite automata on infinite words are called ***ω-regular languages*.**

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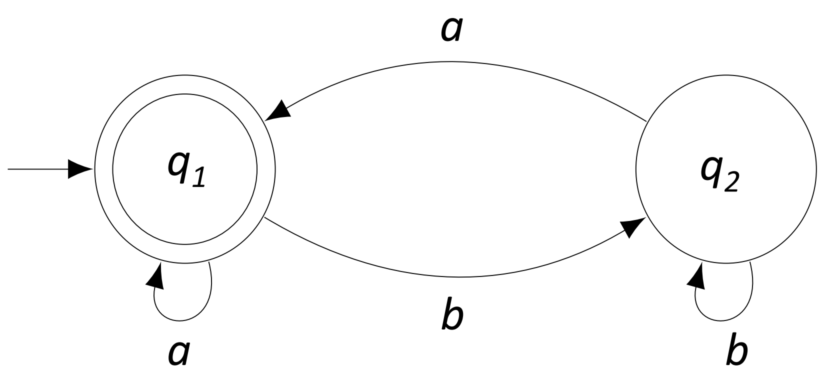
ToDo

- What is the language of this automaton?



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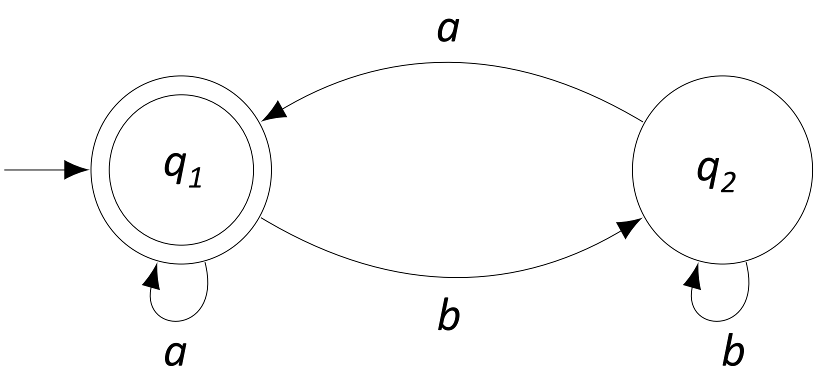

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$\mathcal{L}(\mathcal{B}) = (\{a,b\}^* a)^\omega$

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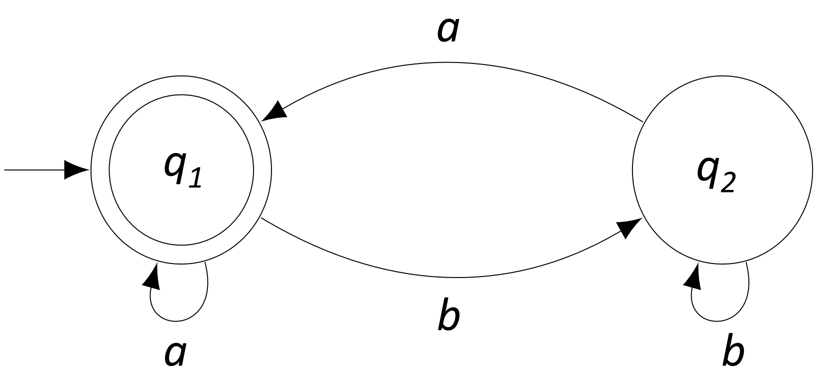
ToDo Can you express it in LTL?



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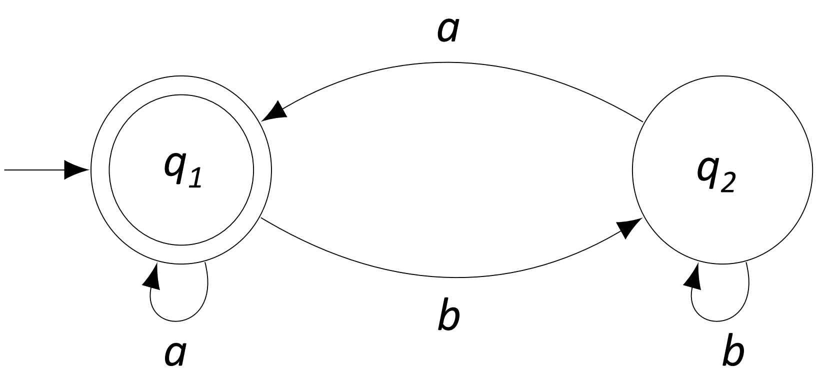

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In LTL: *GF(a)*

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Det. and Non-det. Büchi Automata

- Deterministic Büchi automata are strictly less expressive than nondeterministic ones.
  - That is, not every nondeterministic Büchi automaton has an equivalent deterministic Büchi one.

Det. and Non-det. Büchi Automata

Lemma 1: Let $\mathcal{B}$ be a deterministic Büchi automaton. Then,

$\forall$ w $\in \Sigma^\omega$, w $\in \mathcal{L}(\mathcal{B}) \Leftrightarrow \exists$ infinitely many finite prefixes of w on which $\mathcal{B}$ reaches an accepting state.

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Proof sketch:
A deterministic automaton $\mathcal{B}$ has exactly one run on each word w.
This run is accepting $\Leftrightarrow$
a state $q \in F$ is reached an infinite number of times $\Leftrightarrow$
on infinitely many prefixes of w, q is reached.

□

Det. and Non-det. Büchi Automata

Theorem: There exists a non-deterministic Büchi automaton $\mathcal{B}$ for which there is no equivalent deterministic one.

Det. and Non-det. Büchi Automata

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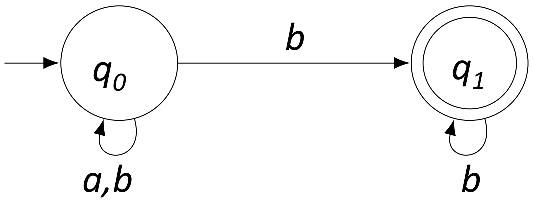
Proof: Consider $\mathcal{B}$ below. What is its language? (Also in LTL)

$\mathcal{B}$



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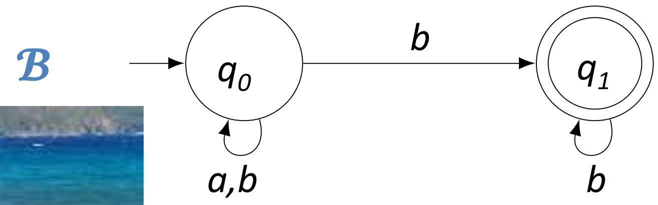


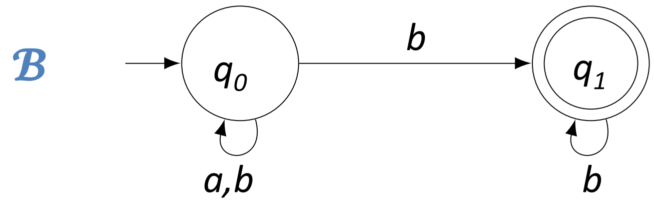
$\mathcal{L}(\mathcal{B})$ = {words with a finite number of a's}
or
$\mathcal{L}(\mathcal{B}) = \{a,b\}^* b^\omega$

In LTL :
FG¬a or **FG**b

Det. and Non-det. Büchi Automata

Proof:

Consider $\mathcal{B}$ below. $\mathcal{L}(\mathcal{B})$={words with finitely many a's}. Assume there exists deterministic $\mathcal{C}$: $\mathcal{L}(\mathcal{C}) = \mathcal{L}(\mathcal{B})$. Note that for any finite word σ, $\sigma \cdot b^{\omega} \in \mathcal{L}(\mathcal{B})$, so $\mathcal{C}$ accepts $\sigma \cdot b^{\omega}$ as well.



Det. and Non-det. Büchi Automata

- Since $b^{\omega} \in \mathcal{L}(C)$, there is an accepting state $q_1 \in F$, so that the run on b^{ω} passes through q_1 after b^{n1}

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- Since $b^{n1}ab^\omega \in \mathcal{L}(C)$, there is $q_2 \in \mathbf{F}$, so that the run on $b^{n1}ab^\omega$ passes through q_2 after $b^{n1}ab^{n2}$

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- And so on…

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- And so on…

- For every k, there is $q_k \in \mathbf{F}$, reached after $b^{n1}ab^{n2}\ldots ab^{nk}$

Det. and Non-det. Büchi Automata

- Since $\mathcal{C}$ is deterministic, there is one run going through $q_1\ q_2\ q_3 \ldots q_k$

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- Thus, in $\mathcal{C}$ there is a lasso-shape run that goes through q_j infinitely often, passing through a infinitely often.

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- This run is accepting a word with infinitely many a.

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- This run is accepting a word with infinitely many a.
- A contradiction!

- Conclusion: there is no det. automaton, equivalent to $\mathcal{B}$. □

Det. and Non-det. Büchi Automata

Lemma 2: Deterministic Büchi automata are not closed under complementation.

ToDo

Proof Idea for Lemma 2?

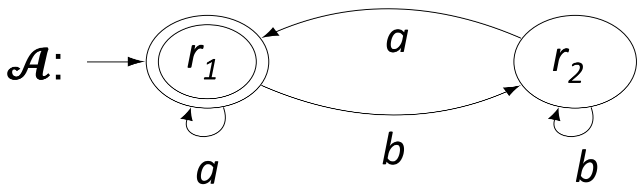
- Hint: Consider the language $\mathcal{L}(\mathcal{A})$ = {words with infinitely many a's}.

Det. and Non-det. Büchi Automata

Lemma 2: Deterministic Büchi automata are not closed under complementation.

Proof:

- Consider the language $\mathcal{L}$ = {words with infinitely many a's}.
- Construct a deterministic Büchi automaton $\mathcal{A}$ that accepts $\mathcal{L}$.

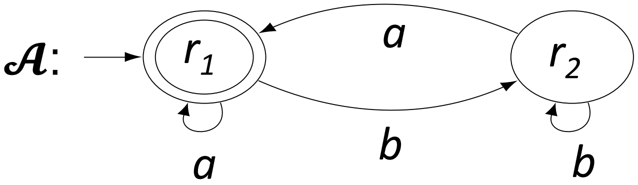


Det. and Non-det. Büchi Automata

Lemma 2: Deterministic Büchi automata are not closed under complementation.

Proof:

- Consider the language $\mathcal{L}$ = {words with infinitely many a's}.
- Construct a deterministic Büchi automaton $\mathcal{A}$ that accepts $\mathcal{L}$.
- Its complement is $\mathcal{L}$'={words with finitely many a's}, for which there is no deterministic Büchi automaton (see Theorem). □



Det. and Non-det. Büchi Automata

Theorem: Nondeterministic Büchi automata are closed under complementation.

- The construction is very complicated. We will not see it here.
- Originally Büchi showed an algorithm for complementation that is double exponential in the size n of the automaton
- Mooly Safra (Tel-Aviv University) proved that it can be done by

$$2^{O(n \log n)}$$

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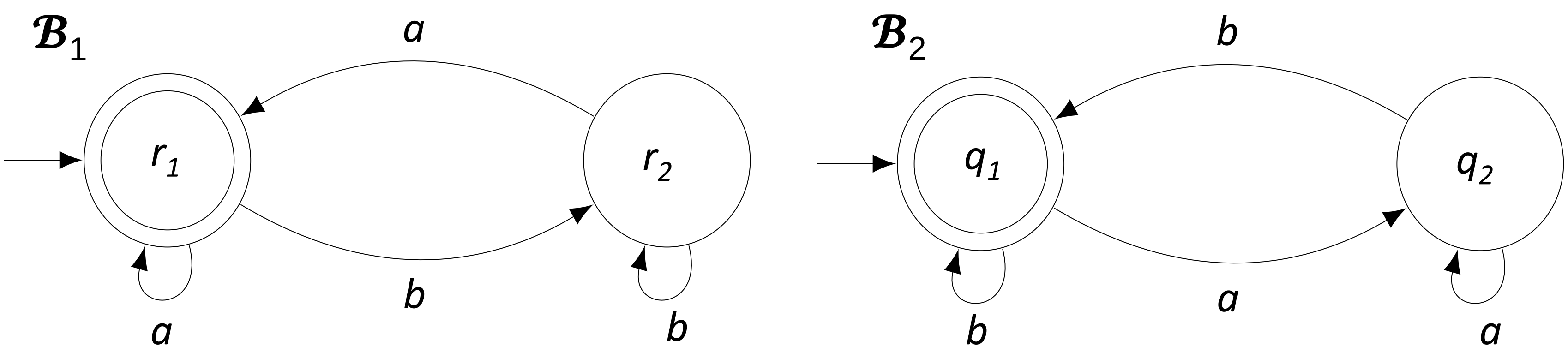
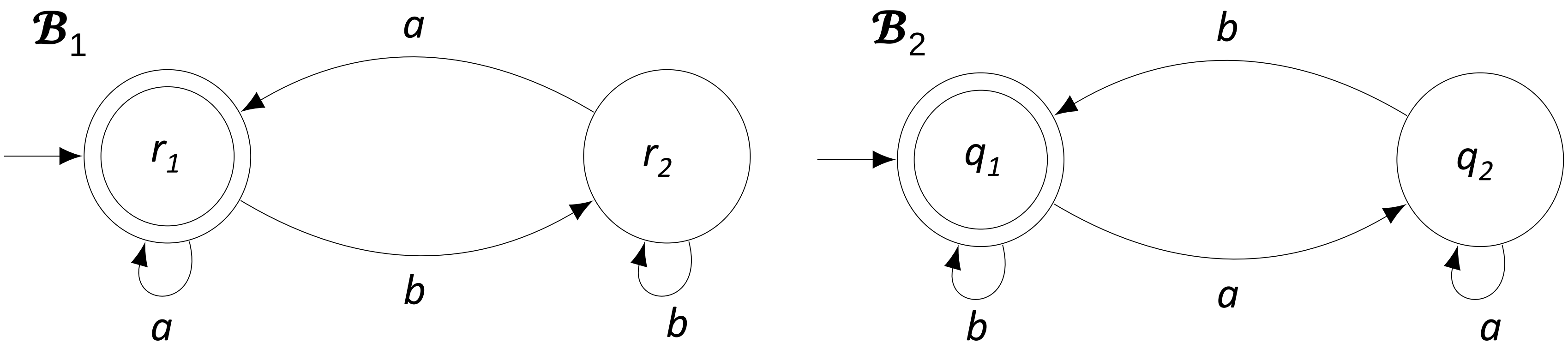
Intersection of Büchi Automata



- What is $\mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$?
- The language $\mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$ =
  {words with an infinite number of a's and infinite number of b's} - not empty

Intersection of Büchi Automata



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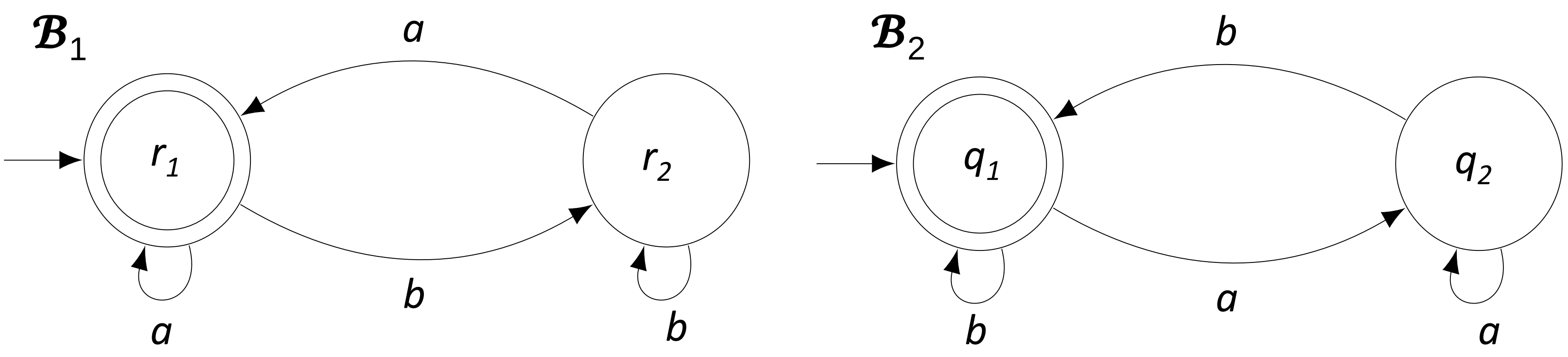

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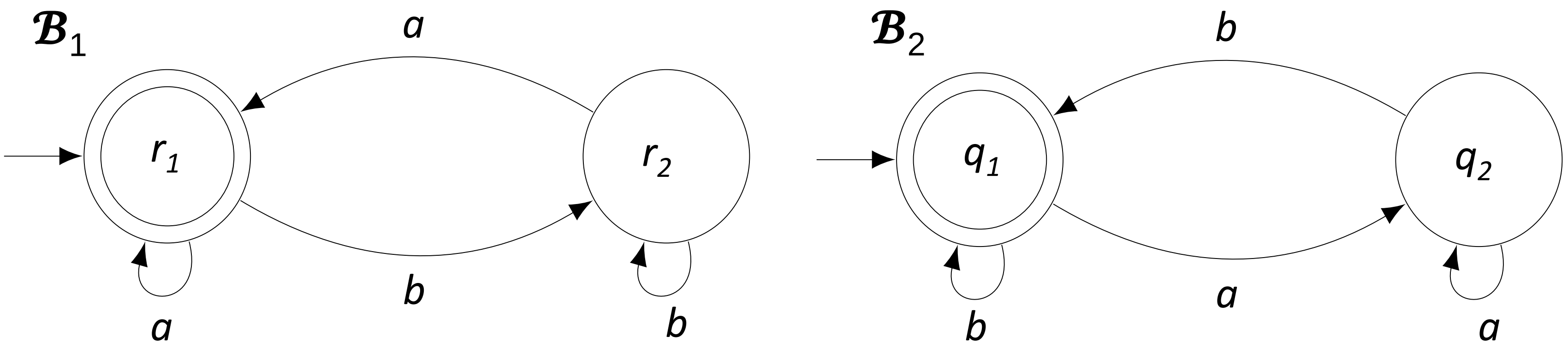
ToDo

- What do you get if you build the standard intersection?

Intersection of Büchi Automata



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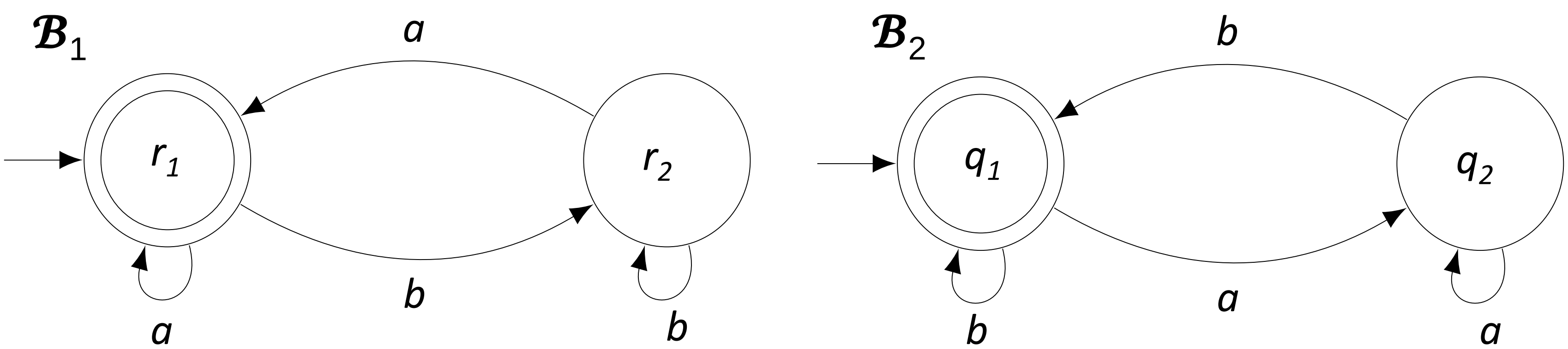
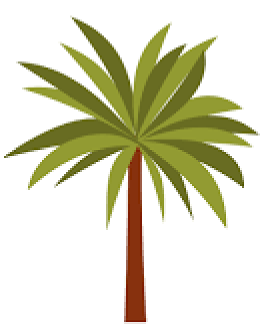


- $\mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$ =
  {words with an infinite number of a's and infinite number of b's}
- A standard intersection does not work – the automaton will not have any accepting states!



Intersection of Büchi Automata

- Given $\mathcal{B}_1 = (\Sigma, Q_1, \Delta_1, Q_1^0, F_1)$ and $\mathcal{B}_2 = (\Sigma, Q_2, \Delta_2, Q_2^0, F_2)$
- $\mathcal{B} = (\Sigma, Q, \Delta, Q^0, F)$ s.t. $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$ is defined as follows:
  - $Q = Q_1 \times Q_2 \times \{0, 1, 2\}$
  - $Q^0 = Q_1^0 \times Q_2^0 \times \{0\}$
  - $F = Q_1 \times Q_2 \times \{2\}$

Intersection of Büchi Automata

$((q_1,q_2,x), a, (q'_1,q'_2,x')) \in \mathbf{\Delta} \Leftrightarrow$

(1) $(q_1,a,q'_1) \in \mathbf{\Delta}_1$ and $(q_2,a,q'_2) \in \mathbf{\Delta}_2$ and

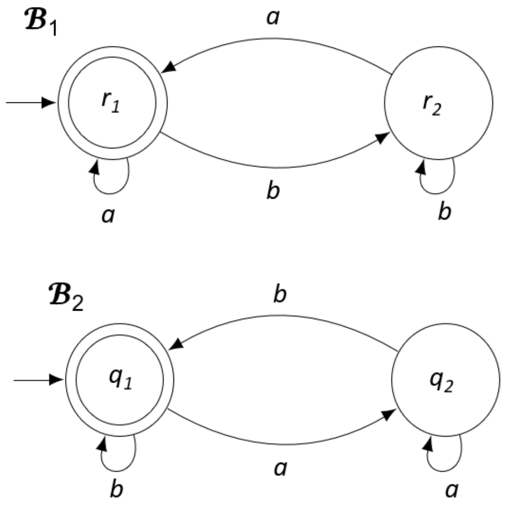
(2) If $x=0$ and $q'_1 \in \mathbf{F}_1$ then $x'=1$
If $x=1$ and $q'_2 \in \mathbf{F}_2$ then $x'=2$
If $x=2$ then $x'=0$
Else, $x'=x$

Explanation: $x=0$ is waiting for an accepting state from $\mathbf{F}_1$

$x=1$ is waiting for an accepting state from $\mathbf{F}_2$

Intersection of Büchi Automata

- The first copy waits for an accepting state of $\mathcal{B}_1$
- The second copy waits for an accepting state of $\mathcal{B}_2$
- All states in the third copy are accepting
- Only the reachable states are drawn



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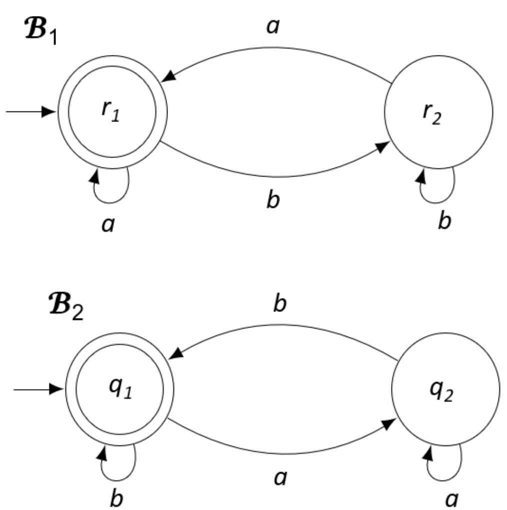

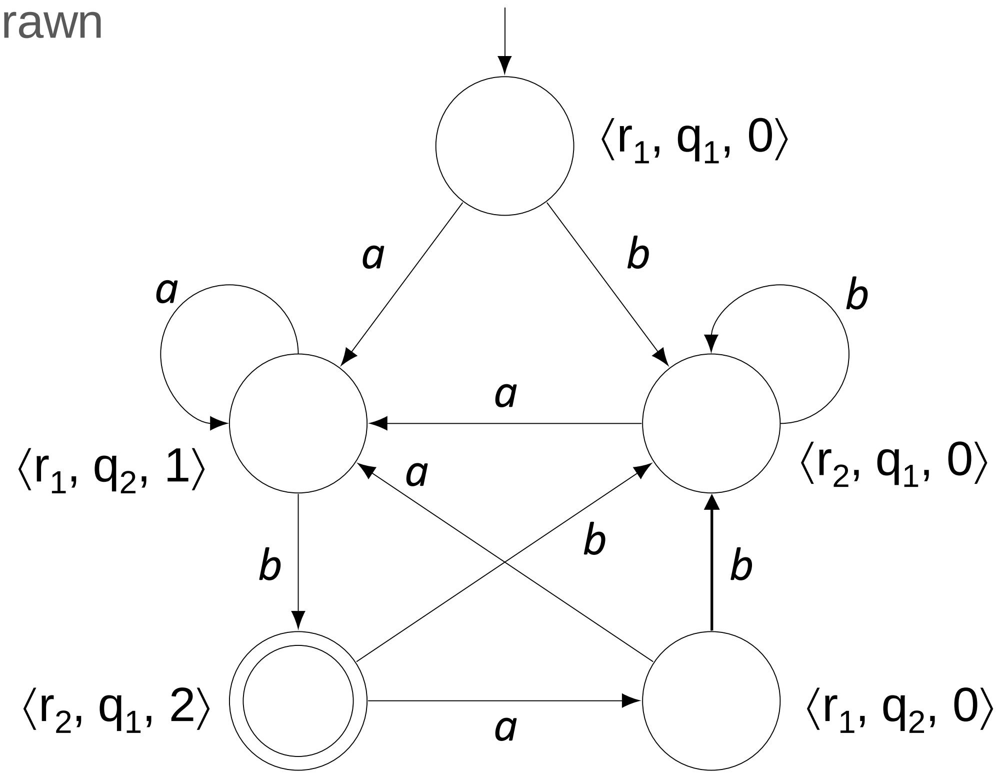


THANK
YOU!